

Numerical Approximation of a Boundary Value Problem with Superlinear Non-linearity on the Boundary

Larissa Renshaw Shalmali Bandyopadhyay Thomas L. Lewis

Department of Mathematics & Statistics, University of Tennessee at Martin

Problem

We consider the nonlinear boundary value problem below:

$$\begin{cases} u'' + u = 0 & \text{in } (a, b) \\ u'(a) = u'(b) = \lambda f(v) \\ v'' + v = 0 & \text{in } (a, b) \\ v'(a) = v'(b) = \lambda f(u) \end{cases} \quad (\text{DE})$$

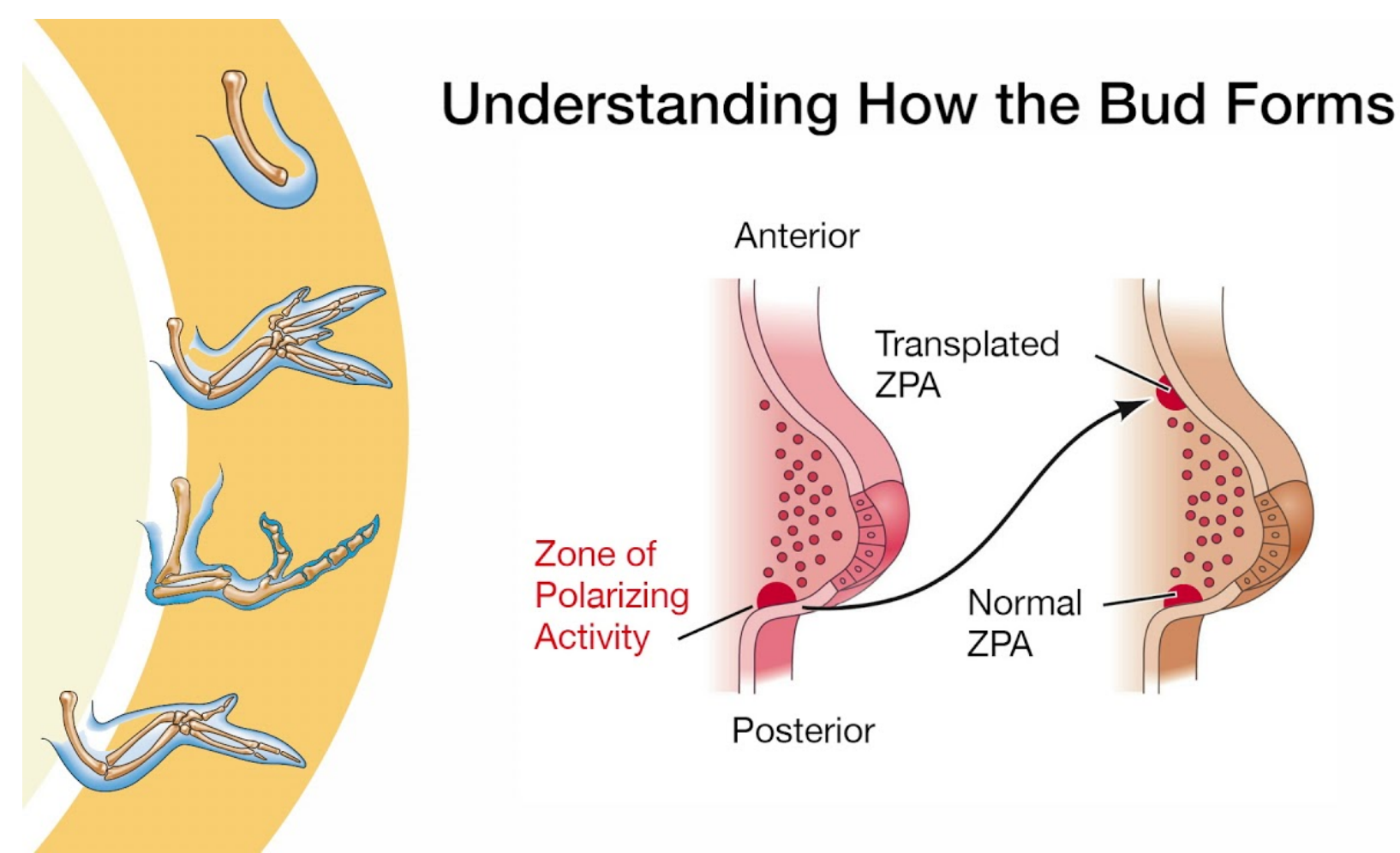
where

- λ is a parameter,
- f is superlinear at infinity, and
- $f(0) = 0$.

Simple examples of $f(s)$:

$$\begin{aligned} f(s) &= s^2 \\ f(s) &= s^2 + s \\ f(s) &= s^2 + 2s \\ f(s) &= s^3 - 0.1s^2 + 0.1s \end{aligned}$$

Motivation



Limb development which incorporates both outgrowth due to cell growth as well as cell division.

H. O. R. Dillon A mathematical model for outgrowth and spatial patterning of the vertebrate limb bud *J. Theoret. Biol.*, 197:295-300, 1999.

Theoretical Results Obtained

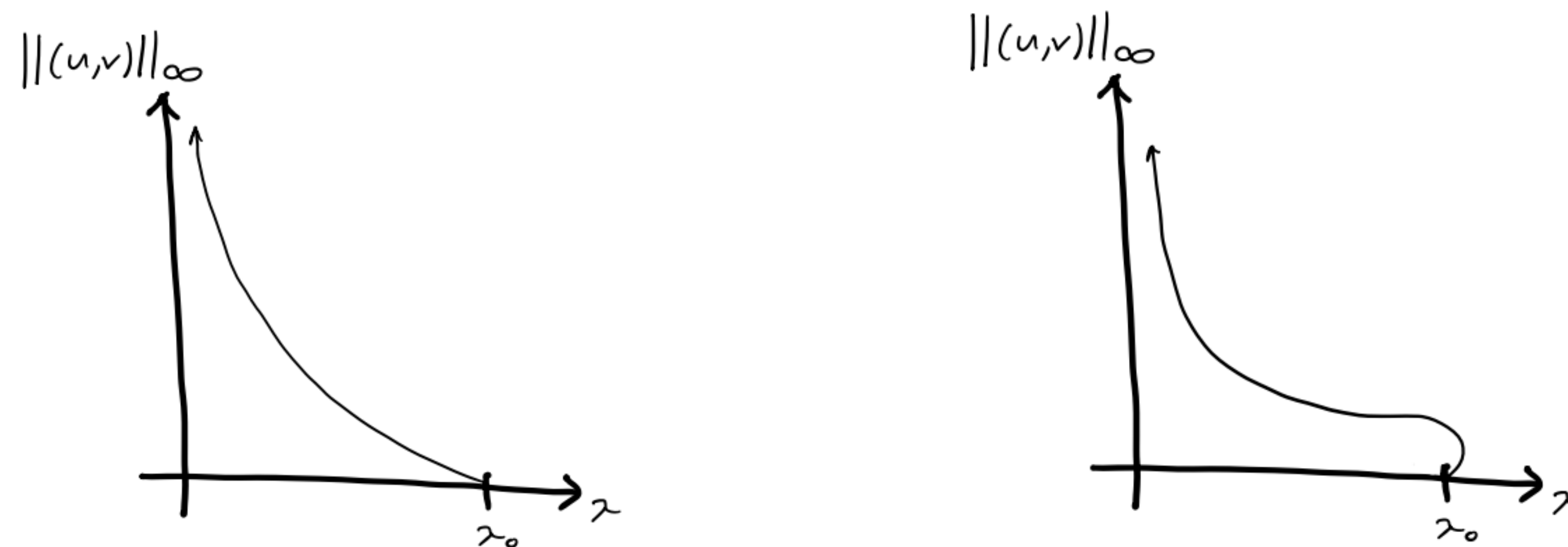
Connected component of positive weak solution of (DE) bifurcating from the trivial solution at λ_0 also bifurcates from infinity at 0.

[1] Bandyopadhyay, S . and Chhetri, M . and Delgado, B . B . and Mavinga, N . and Pardo, R . Bifurcation and Multiplicity Results for Elliptic Problems with Subcritical Nonlinearity on the Boundary submitted, 2021.

Selected References

- [1] Bandyopadhyay, Shalmali (2022). *Solvability of Classes of Elliptic Problems with Nonlinear Boundary Conditions* (dissertation).
- [2] D. B. Gurung, K. C. Gokul, and P.R. Adhikary. Mathematical model of thermal effects of blinking in human eye *Int. J. Biomath*, 9(1):1650006, 24, 2016.
- [3] Bandyopadhyay, S . and Chhetri, M . and Delgado, B . B . and Mavinga, N . and Pardo, R . Maximal and minimal weak solutions for elliptic problems with nonlinearity on the boundary *Electron. Res. Arch.*, 30(6):2121–2137, 2022.

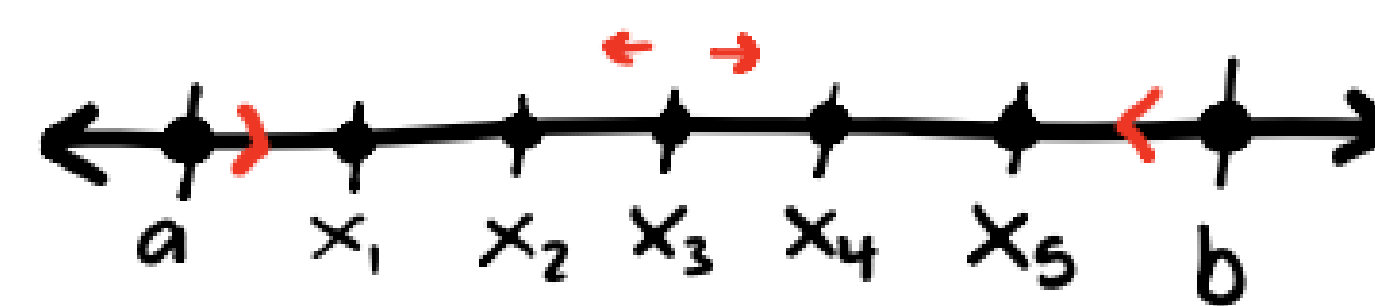
Expected Bifurcation Diagrams



- $\|u\|_\infty = \sup_{a \leq x \leq b} \{|u(x)|\}$
- $\|v\|_\infty = \sup_{a \leq x \leq b} \{|v(x)|\}$
- $\|(u, v)\|_\infty = \|u\|_\infty + \|v\|_\infty$

Construction of Nodes

- $N = 1$
- $\Omega = [a, b]$
- $h = x_k - x_{k-1}$



First-order Discrete

Forward and backward difference operators, respectively.

$$D_h^+ f(x) \equiv \frac{f(x+h) - f(x)}{h}, \quad h > 0$$

$$D_h^- f(x) \equiv \frac{f(x) - f(x-h)}{h}, \quad h > 0$$

where the accuracy is $\mathcal{O}(h)$.

Second-order Discrete

Second-order difference operator.

$$D_h^2 f(x) \equiv \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}, \quad h > 0$$

where the accuracy is $\mathcal{O}(h^2)$.

Bifurcation Diagrams (Single Equation Case)

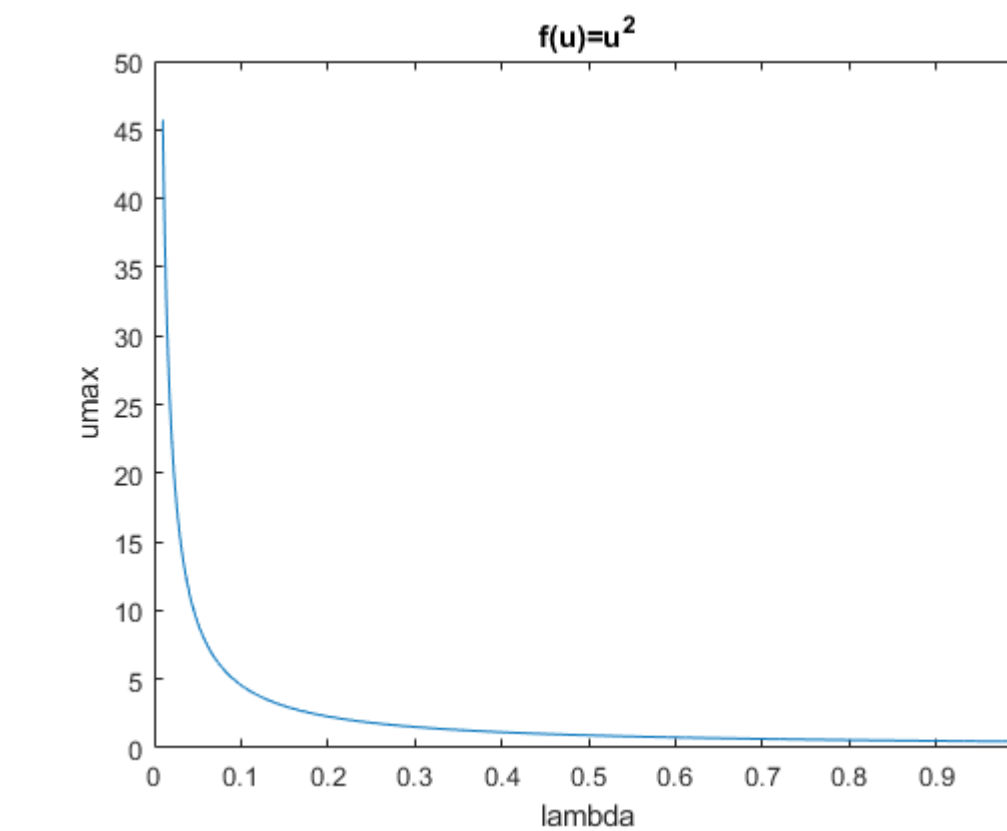


Figure 1: $f(s) = s^2$

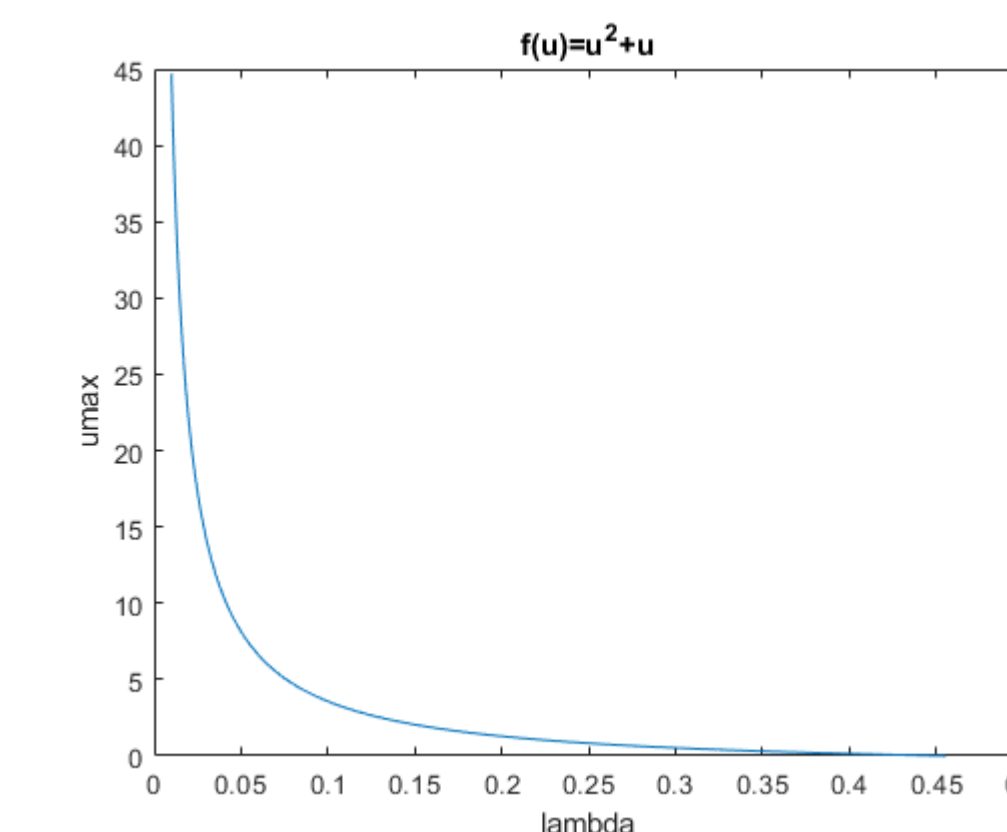


Figure 2: $f(s) = s^2 + s$

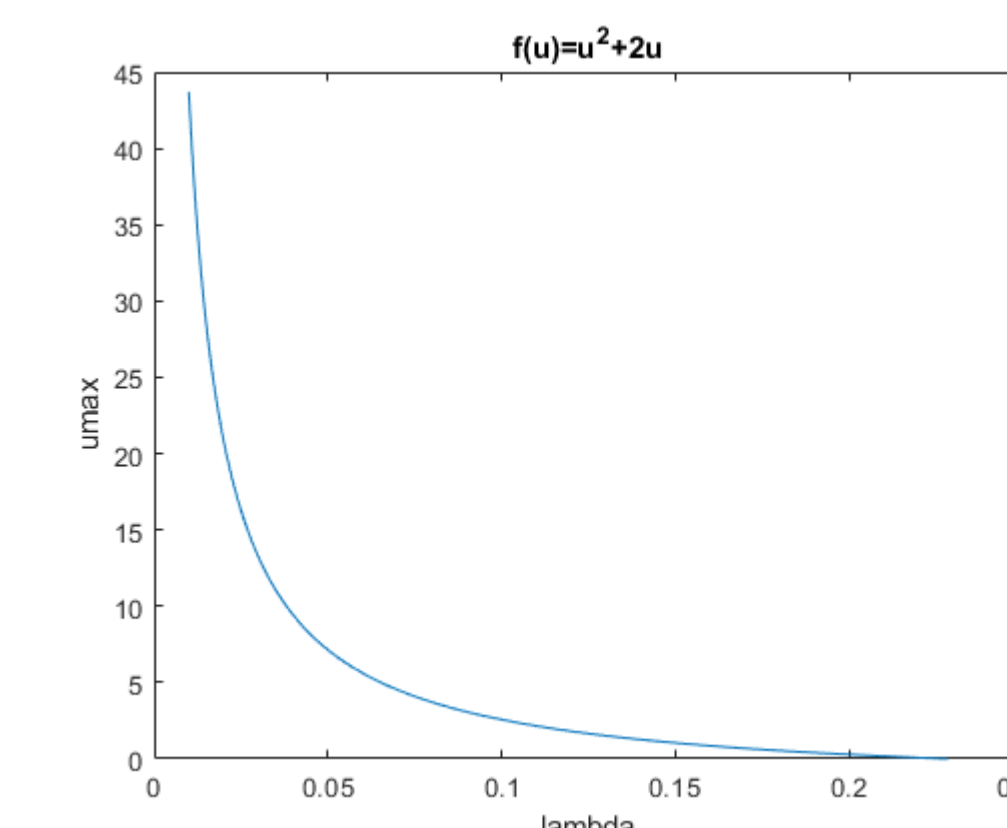


Figure 3: $f(s) = s^2 + 2s$

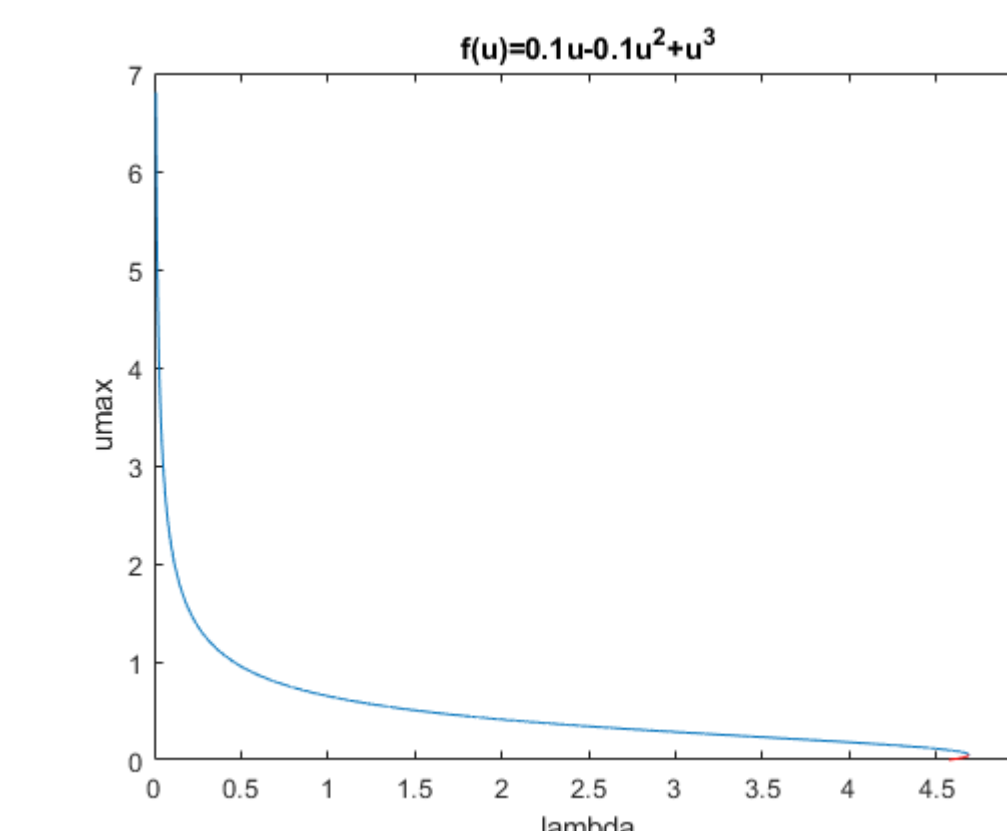


Figure 4: $f(s) = s^3 - 0.1s^2 + 0.1s$

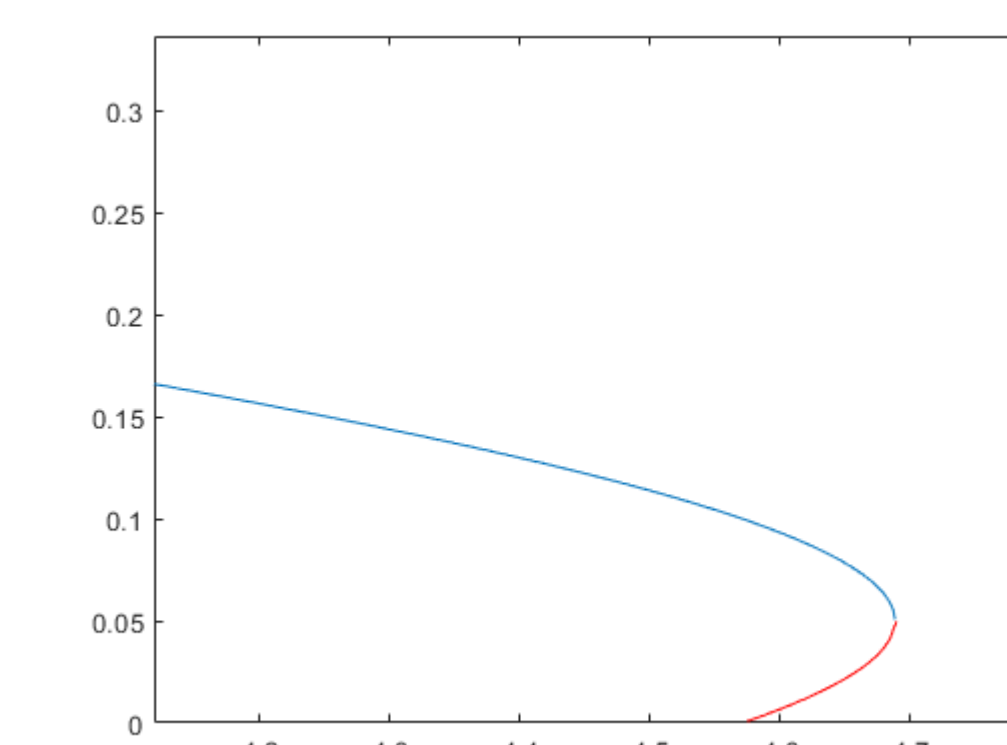


Figure 5: Magnification of turning point in Figure 4