FUNCTION SPACES

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1. What is a Function Space?

Definition 1.1. Let $[a, b]$ be a closed bounded interval. We denote by $\mathbb{R}[a, b]$ the set of all real-valued functions defined on $[a, b]$; that is,

$$\mathbb{R}[a, b] = \{f : [a, b] \to \mathbb{R}\}.$$

Similarly, $\mathbb{C}[a, b]$ denotes the set of all complex-valued functions on $[a, b]$.

Remark. The notation $\mathbb{R}[a, b]$ is local to these notes and should not be confused with the polynomial ring $\mathbb{R}[x]$ from algebra. Here it always means the set of real-valued functions on the interval $[a, b]$.

Definition 1.2. For $f, g \in \mathbb{R}[a, b]$ and $\alpha \in \mathbb{R}$, define the functions $f + g$ and αf by

$$(f + g)(x) := f(x) + g(x), \qquad (\alpha f)(x) := \alpha \cdot f(x), \qquad x \in [a, b].$$

The *zero function* $\mathbf{0} \in \mathbb{R}[a, b]$ is defined by $\mathbf{0}(x) = 0$ for all $x \in [a, b]$.

Proposition 1.3. *With these operations, $\mathbb{R}[a, b]$ is closed under addition and scalar multiplication.*

Proof. Let $f, g \in \mathbb{R}[a, b]$ and $\alpha \in \mathbb{R}$. For every $x \in [a, b]$, both $f(x)$ and $g(x)$ are real numbers, so $f(x) + g(x)$ is a real number. Thus $f + g : [a, b] \to \mathbb{R}$ is a well-defined real-valued function, hence $f + g \in \mathbb{R}[a, b]$. Similarly, $\alpha \cdot f(x)$ is a real number for every $x \in [a, b]$, so $\alpha f \in \mathbb{R}[a, b]$. $\square$

Exercise 1.4. Verify that $\mathbb{R}[a, b]$ with the operations above is a vector space over $\mathbb{R}$. That is, check the eight vector space axioms: associativity and commutativity of addition, existence of an additive identity and inverses, compatibility of scalar multiplication, the scalar identity, and the two distributive laws.

Theorem 1.5 (Subspace Test)**.** *A subset $V \subseteq \mathbb{R}[a, b]$ is a vector subspace if and only if*

(i) $\mathbf{0} \in V$,
(ii) $f + g \in V$ *for all* $f, g \in V$, *and*
(iii) $\alpha f \in V$ *for all* $f \in V$ *and all* $\alpha \in \mathbb{R}$.

Definition 1.6. A subset $V \subseteq \mathbb{R}[a, b]$ that satisfies the three conditions of Theorem 1.5 is called a *function space* on $[a, b]$.

Example 1.7. For each integer $n \geq 0$, the set

$$\mathcal{P}_n[a, b] := \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n : a_0, a_1, \ldots, a_n \in \mathbb{R}\}$$

of polynomials of degree at most n is a function space on $[a, b]$.

Proof. We verify the three conditions of Theorem 1.5.

(i) Taking $a_0 = a_1 = \cdots = a_n = 0$ gives the zero function, so $\mathbf{0} \in \mathcal{P}_n[a, b]$.

(ii) Let $f(x) = \sum_{k=0}^{n} a_k x^k$ and $g(x) = \sum_{k=0}^{n} b_k x^k$ be in $\mathcal{P}_n[a, b]$. Then

$$(f + g)(x) = \sum_{k=0}^{n} (a_k + b_k) x^k,$$

which is a polynomial of degree at most n. Hence $f + g \in \mathcal{P}_n[a, b]$.

(iii) Let $f \in \mathcal{P}_n[a, b]$ as above and $\alpha \in \mathbb{R}$. Then

$$(\alpha f)(x) = \sum_{k=0}^{n} (\alpha a_k) x^k \in \mathcal{P}_n[a, b].$$ $\square$

Exercise 1.8. Show that the functions $1, x, x^2, \ldots, x^n$ form a basis for $\mathcal{P}_n[a, b]$, and conclude that

$$\dim \mathcal{P}_n[a, b] = n + 1.$$

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Example 1.9. The set $\mathcal{P}[a,b]$ of all polynomials (of any degree) on $[a,b]$ is a function space.

Proof. The zero polynomial is in $\mathcal{P}[a,b]$, so condition (i) holds. The sum of two polynomials is a polynomial, and a scalar multiple of a polynomial is a polynomial, so conditions (ii) and (iii) hold. □

Theorem 1.10. *$\mathcal{P}[a,b]$ is infinite-dimensional. That is, for every integer $n \geq 0$, there exist $n+1$ linearly independent functions in $C[a,b]$.*

Proof. Fix $n \geq 0$. The functions $1, x, x^2, \ldots, x^n$ all belong to $\mathcal{P}[a,b]$, and they are linearly independent over $\mathbb{R}$: if

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = 0 \quad \text{for all } x \in [a,b],$$

then since a nonzero polynomial of degree at most n can have at most n roots, but the equation holds on the entire interval $[a,b]$ (uncountably many points), we must have $c_0 = c_1 = \cdots = c_n = 0$. Thus $\mathcal{P}[a,b]$ contains $n+1$ linearly independent elements for every $n \geq 0$, and so $\mathcal{P}[a,b]$ cannot be spanned by any finite set. □

Exercise 1.11. Let $\mathcal{R}$ denote the set of all real rational functions, that is, expressions of the form $r(x) = p(x)/q(x)$ where p and q are polynomials and q is not identically zero. Explain why $\mathcal{R}$ is not a function space on $[a,b]$.
Hint: where is $r(x) = 1/x$ defined?

Definition 1.12. The set of continuous real-valued functions on $[a,b]$ is denoted

$$C[a,b] := \{f \in \mathbb{R}[a,b] : f \text{ is continuous on } [a,b]\}.$$

Proposition 1.13. *$C[a,b]$ is a function space on $[a,b]$.*

Proof. We verify the three conditions of Theorem 1.5.
(i) The zero function is continuous on $[a,b]$, so $\mathbf{0} \in C[a,b]$.
(ii) If $f, g \in C[a,b]$, then $f+g$ is continuous on $[a,b]$, since the sum of two continuous functions is continuous.
(iii) If $f \in C[a,b]$ and $\alpha \in \mathbb{R}$, then αf is continuous on $[a,b]$. □

Since every polynomial is continuous, $\mathcal{P}[a,b] \subseteq C[a,b]$, and so $\mathcal{P}[a,b]$ is a subspace of $C[a,b]$.

Exercise 1.14. Try to find a basis for $C[a,b]$. What is $\dim C[a,b]$?

The exercise above has no satisfactory answer: there is no finite basis for $C[a,b]$.

Exercise 1.15. $C[a,b]$ is infinite-dimensional.

Remark. This is the first essential difference between $\mathcal{P}_n[a,b]$ and $C[a,b]$. The space $\mathcal{P}_n[a,b]$ behaves like the familiar finite-dimensional vector spaces from linear algebra: it has a finite basis, a well-defined dimension, and every linear operator on it has a matrix representation. The space $C[a,b]$ has none of these properties in the finite sense. To work with $C[a,b]$ effectively, one must replace tools from linear algebra with tools from analysis, including norms, convergence, and limits.

The subspace $\mathcal{P}[a,b]$ of polynomials is strictly smaller than $C[a,b]$ (the function $\sin x$ is continuous but is not a polynomial), yet every continuous function on $[a,b]$ can be approximated arbitrarily well by polynomials in the uniform sense. This is the Weierstrass approximation theorem, which we will revisit later. The distinction between a subspace and a dense subspace has no analog in finite dimensions.

Definition 1.16. The set of differentiable real-valued functions on $[a,b]$ is denoted

$$D[a,b] := \{f \in \mathbb{R}[a,b] : f \text{ is differentiable on } [a,b]\}.$$

Proposition 1.17. *$D[a,b]$ is a function space on $[a,b]$.*

Proof. The zero function is differentiable, and the sum and scalar multiple of differentiable functions are differentiable. The conditions of Theorem 1.5 hold. □

Lemma 1.18. *$D[a,b] \subseteq C[a,b]$. That is, every differentiable function on $[a,b]$ is continuous.*

Proof. Let $f \in D[a,b]$ and fix $c \in [a,b]$. We show f is continuous at c.

Since f is differentiable at c, the limit

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists as a real number. For $x \neq c$, write

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c} \cdot (x - c).$$

Taking the limit as $x \to c$:

$$\lim_{x\to c}\big(f(x)-f(c)\big) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}\cdot\lim_{x\to c}(x-c) = f'(c)\cdot 0 = 0.$$

Hence $\lim_{x\to c} f(x) = f(c)$, so f is continuous at c. Since $c \in [a,b]$ was arbitrary, f is continuous on $[a,b]$. □

Corollary 1.19. *$D[a,b]$ is a vector subspace of $C[a,b]$.*

Definition 1.20. For each integer $k \geq 0$, define

$$C^k[a,b] := \{f \in \mathbb{R}[a,b] : f \text{ has } k \text{ continuous derivatives on } [a,b]\}.$$

By convention $C^0[a,b] := C[a,b]$. Define also

$$C^\infty[a,b] := \bigcap_{k=0}^{\infty} C^k[a,b] = \{f \in \mathbb{R}[a,b] : f \text{ has derivatives of all orders on } [a,b]\}.$$

Elements of $C^\infty[a,b]$ are called *smooth* functions.

Proposition 1.21. *For each $k \geq 0$, the set $C^k[a,b]$ is a function space on $[a,b]$. The set $C^\infty[a,b]$ is also a function space.*

Proof. For each k, the zero function has continuous derivatives of all orders, so $\mathbf{0} \in C^k[a,b]$. If $f, g \in C^k[a,b]$, then $(f+g)^{(k)} = f^{(k)} + g^{(k)}$ is a sum of continuous functions, hence continuous; the same holds for all lower-order derivatives. Hence $f + g \in C^k[a,b]$. Similarly $(\alpha f)^{(k)} = \alpha f^{(k)}$ is continuous, so $\alpha f \in C^k[a,b]$.

The intersection of vector subspaces is a vector subspace. Since each $C^k[a,b]$ is a vector subspace of $\mathbb{R}[a,b]$, so is their intersection. □

Theorem 1.22. *For every $k \geq 1$, $C^k[a,b] \subseteq C^{k-1}[a,b]$. Hence*

$$C^\infty[a,b] \subseteq \cdots \subseteq C^k[a,b] \subseteq C^{k-1}[a,b] \subseteq \cdots \subseteq C^1[a,b] \subseteq C[a,b] \subseteq \mathbb{R}[a,b].$$

Proof. Let $f \in C^k[a,b]$. By definition, f has k continuous derivatives on $[a,b]$. In particular, f has $k-1$ continuous derivatives, so $f \in C^{k-1}[a,b]$. □

Remark. The inclusions in Theorem 1.22 are strict for each $k \geq 1$ on a nondegenerate interval $[a,b]$. The function $f(x) = |x-c|$ for any $c \in (a,b)$ belongs to $C[a,b]$ but not to $C^1[a,b]$,

Exercise 1.23. Find a function that belongs to $C^1[a,b]$ but not to $C^2[a,b]$.

2. L^p Spaces

In the discovery activity, we observed that there is no single notion of "size" for a function. Different measurement schemes capture different features: peak height, total area, or something in between. We now make this precise. The family of L^p spaces, indexed by a real parameter $p \in [1,\infty]$, provides a natural family of size measurements interpolating between area ($p = 1$) and peak ($p = \infty$).

Measure Zero and Almost Everywhere. Before defining the L^p norm, we need a notion that will appear repeatedly: a property holding "almost everywhere" on $[a,b]$. Two functions can differ at a small set of points without affecting their integrals, and we want this reflected in our definitions.

Definition 2.1. A set $E \subseteq [a,b]$ has *measure zero* if for every $\epsilon > 0$, there exist countably many open intervals (c_n, d_n) such that

$$E \subseteq \bigcup_{n=1}^{\infty}(c_n,d_n) \quad\text{and}\quad \sum_{n=1}^{\infty}(d_n - c_n) < \epsilon.$$

Example 2.2. Every finite set has measure zero: cover $\{x_1,\ldots,x_N\}$ by N intervals each of length ϵ/N.

Example 2.3. Every countable set has measure zero: enumerate $E = \{x_1, x_2, x_3, \ldots\}$ and cover x_n by an interval of length $\epsilon/2^n$. The total length is $\sum_n \epsilon/2^n = \epsilon$. In particular, $\mathbb{Q} \cap [a,b]$ has measure zero.

Definition 2.4. A property $P(x)$ holds *almost everywhere* (abbreviated *a.e.*) on $[a,b]$ if the set $\{x \in [a,b] : P(x) \text{ fails}\}$ has measure zero.

The reason measure-zero sets are negligible for our purposes is that they do not contribute to integrals. If $f, g : [a, b] \to \mathbb{R}$ are bounded and $f(x) = g(x)$ for every x outside a set of measure zero, then f and g have the same integral:

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx.$$

The intuition is that the Riemann sum for $\int f$ approximates the area under the graph of f by summing $f(x_i) \cdot \Delta x_i$ over a partition; if f and g differ only on a set that can be covered by intervals of arbitrarily small total length, those discrepancies contribute arbitrarily little to the Riemann sums.

In particular, modifying a function at finitely or countably many points does not change its integral. Two functions that agree a.e. are interchangeable from the standpoint of integration.

The L^p Norm and the L^p Spaces.

Definition 2.5. Let $1 \le p < \infty$. For a function $f : [a, b] \to \mathbb{R}$, define the L^p *norm* of f by

$$\|f\|_p := \left(\int_a^b |f(x)|^p\,dx \right)^{1/p}.$$

The L^p *space* is

$$L^p[a, b] := \{ f : [a, b] \to \mathbb{R} \mid \|f\|_p < \infty \},$$

where functions equal almost everywhere are identified as the same element.

Definition 2.6. The *essential supremum* of $f : [a, b] \to \mathbb{R}$ is

$$\|f\|_\infty := \inf\{ M \ge 0 : |f(x)| \le M \text{ for almost every } x \in [a, b] \}.$$

The space $L^\infty[a, b]$ consists of all f with $\|f\|_\infty < \infty$, again with the identification of functions equal almost everywhere.

Remark. The identification of functions equal almost everywhere is forced by the definitions. If f and g differ on a set of measure zero, then $|f - g|^p$ has integral zero, and so $\|f - g\|_p = 0$. Without the identification, $\|\cdot\|_p$ would not satisfy the first norm axiom below. With the identification, elements of L^p are equivalence classes; we continue to write functions for elements, with the understanding that each represents a class.

Norm Axioms. A function $\|\cdot\| : V \to [0, \infty)$ on a vector space V is called a *norm* if it satisfies, for all $f, g \in V$ and all $\alpha \in \mathbb{R}$:

(N1) **(Positivity)** $\|f\| \ge 0$, with $\|f\| = 0$ if and only if $f = 0$ (in V).
(N2) **(Homogeneity)** $\|\alpha f\| = |\alpha| \cdot \|f\|$.
(N3) **(Triangle inequality)** $\|f + g\| \le \|f\| + \|g\|$.

Proposition 2.7. *For $1 \le p < \infty$, the function $\|\cdot\|_p$ satisfies axioms (N1) and (N2).*

Proof. **(N1)** The integrand $|f(x)|^p \ge 0$, so $\int |f|^p \ge 0$ and $\|f\|_p \ge 0$. If $\|f\|_p = 0$, then $\int |f|^p = 0$. Since $|f|^p \ge 0$, this forces $|f(x)|^p = 0$ for almost every x, hence $f = 0$ almost everywhere; in L^p, this is the zero element.

(N2) Compute directly:

$$\|\alpha f\|_p = \left(\int |\alpha f|^p \right)^{1/p} = \left(|\alpha|^p \int |f|^p \right)^{1/p} = |\alpha| \cdot \|f\|_p. \qquad \square$$

The triangle inequality is more delicate and is the content of Minkowski's inequality, proved below via Hölder's inequality.

Cauchy–Schwarz, Hölder, and Minkowski. We begin with the most fundamental case, which is the inner-product inequality on L^2.

Theorem 2.8 (Cauchy–Schwarz Inequality). *For $f, g \in L^2[a, b]$,*

$$\left| \int_a^b f(x) g(x)\,dx \right| \le \|f\|_2 \cdot \|g\|_2.$$

Proof. If $\|g\|_2 = 0$ then $g = 0$ a.e. and both sides vanish. Assume $\|g\|_2 > 0$ and consider, for $t \in \mathbb{R}$,

$$0 \le \int_a^b \big(f(x) - tg(x)\big)^2 dx = \|f\|_2^2 - 2t \int fg + t^2 \|g\|_2^2.$$

This is a non-negative quadratic in t, so its discriminant is non-positive:

$$4\left(\int fg\right)^2 - 4\|f\|_2^2\|g\|_2^2 \le 0.$$

Rearranging and taking square roots yields the inequality. □

We now generalize to arbitrary conjugate exponents.

Definition 2.9. For $1 < p < \infty$, the *conjugate exponent* q is defined by $\frac{1}{p} + \frac{1}{q} = 1$. Equivalently, $q = p/(p-1)$. By convention, the conjugate of $p = 1$ is $q = \infty$, and the conjugate of $p = \infty$ is $q = 1$.

Exercise 2.10 (Young's Inequality)**.** Let $p, q > 1$ be conjugate exponents and let $a, b \ge 0$. Show that

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

Hint: use the concavity of $\log x$ *applied to the points* a^p *and* b^q *with weights* $1/p$ *and* $1/q$*.*

Theorem 2.11 (Hölder's Inequality)**.** *Let* $1 < p < \infty$ *and let* q *be its conjugate. For* $f \in L^p[a, b]$ *and* $g \in L^q[a, b]$*,*

$$\int_a^b |f(x)g(x)|\,dx \le \|f\|_p \cdot \|g\|_q.$$

Proof. If $\|f\|_p = 0$ or $\|g\|_q = 0$, then $fg = 0$ a.e. and the inequality holds trivially. Suppose $\|f\|_p, \|g\|_q > 0$. Apply Young's inequality (Exercise 2.10) pointwise with

$$A = \frac{|f(x)|}{\|f\|_p}, \qquad B = \frac{|g(x)|}{\|g\|_q}.$$

This gives, for each x,

$$\frac{|f(x)|}{\|f\|_p} \cdot \frac{|g(x)|}{\|g\|_q} \le \frac{1}{p} \cdot \frac{|f(x)|^p}{\|f\|_p^p} + \frac{1}{q} \cdot \frac{|g(x)|^q}{\|g\|_q^q}.$$

Integrating over $[a, b]$:

$$\frac{1}{\|f\|_p\|g\|_q} \int_a^b |fg|\,dx \le \frac{1}{p} \cdot \frac{\|f\|_p^p}{\|f\|_p^p} + \frac{1}{q} \cdot \frac{\|g\|_q^q}{\|g\|_q^q} = \frac{1}{p} + \frac{1}{q} = 1.$$

Multiplying through by $\|f\|_p\|g\|_q$ yields Hölder's inequality. □

Remark. Cauchy–Schwarz is the case $p = q = 2$ of Hölder's inequality.

Theorem 2.12 (Minkowski's Inequality)**.** *Let* $1 \le p < \infty$*. For* $f, g \in L^p[a, b]$*,*

$$\|f + g\|_p \le \|f\|_p + \|g\|_p.$$

Proof. The case $p = 1$ follows from $|f + g| \le |f| + |g|$ pointwise and monotonicity of the integral. Assume $1 < p < \infty$. If $\|f + g\|_p = 0$ the inequality is trivial; otherwise, write

$$|f + g|^p = |f + g| \cdot |f + g|^{p-1} \le (|f| + |g|) \cdot |f + g|^{p-1}.$$

Integrating and splitting the right-hand side:

$$\|f + g\|_p^p \le \int |f| \cdot |f + g|^{p-1}\,dx + \int |g| \cdot |f + g|^{p-1}\,dx.$$

Apply Hölder's inequality to each integral with exponents p and $q = p/(p-1)$:

$$\int |f| \cdot |f + g|^{p-1}\,dx \le \|f\|_p \cdot \left(\int |f + g|^{(p-1)q}\right)^{1/q} = \|f\|_p \cdot \|f + g\|_p^{p/q},$$

using $(p-1)q = p$. The same bound holds with g in place of f. Therefore

$$\|f + g\|_p^p \le (\|f\|_p + \|g\|_p) \cdot \|f + g\|_p^{p/q}.$$

Dividing both sides by $\|f + g\|_p^{p/q}$ and using $p - p/q = 1$, we obtain $\|f + g\|_p \le \|f\|_p + \|g\|_p$. □

Corollary 2.13. *For* $1 \le p \le \infty$*,* $\|\cdot\|_p$ *is a norm on* $L^p[a, b]$*.*

Exercise 2.14. Show that $L^p[a,b]$ is a function space for every $1 \le p \le \infty$. (Hint: use Minkowski's inequality for closure under addition and homogeneity for closure under scalar multiplication.)

Special Cases and Example.

Example 2.15. Returning to the three functions from the discovery activity, defined on $[0,1]$:

$$f(x) = \begin{cases} 100 & 0 \le x \le 0.01 \\ 0 & \text{otherwise}, \end{cases} \qquad g(x) = 1, \quad h(x) = \begin{cases} 10 & 0 \le x \le 0.1 \\ 0 & \text{otherwise}. \end{cases}$$

We compute:

$$\begin{aligned} \|f\|_1 &= \int_0^{0.01} 100\,dx = 1, & \|f\|_2 &= \left(\int_0^{0.01} 100^2\,dx\right)^{1/2} = 10, & \|f\|_\infty &= 100, \\ \|g\|_1 &= \int_0^1 1\,dx = 1, & \|g\|_2 &= 1, & \|g\|_\infty &= 1, \\ \|h\|_1 &= \int_0^{0.1} 10\,dx = 1, & \|h\|_2 &= \left(\int_0^{0.1} 100\,dx\right)^{1/2} = \sqrt{10}, & \|h\|_\infty &= 10. \end{aligned}$$

All three functions agree in the L^1 norm (value 1). In the L^∞ norm they are sharply distinguished $(100, 1, 10)$. In the L^2 norm they fall in between $(10, 1, \sqrt{10})$. As p ranges from 1 to ∞, $\|\cdot\|_p$ interpolates between area and peak.

Inner Product on L^2. Among all the L^p spaces, L^2 is distinguished by carrying an inner product compatible with its norm.

Definition 2.16. For $f, g \in L^2[a,b]$, the L^2 *inner product* is

$$\langle f, g\rangle := \int_a^b f(x)g(x)\,dx.$$

Proposition 2.17. *The L^2 inner product satisfies, for all $f, g, h \in L^2[a,b]$ and $\alpha \in \mathbb{R}$:*

(I1) ***(Symmetry)*** $\langle f, g\rangle = \langle g, f\rangle$.

(I2) ***(Linearity in the first argument)*** $\langle \alpha f + h, g\rangle = \alpha\langle f, g\rangle + \langle h, g\rangle$.

(I3) ***(Positive-definiteness)*** $\langle f, f\rangle \ge 0$, *with equality if and only if* $f = 0$.

Moreover, $\langle f, f\rangle = \|f\|_2^2$.

Proof. Symmetry and linearity follow from the corresponding properties of the integral and of multiplication in $\mathbb{R}$. Positive-definiteness follows from $\langle f, f\rangle = \int f^2 = \|f\|_2^2$ and the (N1) axiom for $\|\cdot\|_2$. □

Remark. The other L^p spaces with $p \ne 2$ do not admit an inner product compatible with their norm. This is because an inner-product norm always satisfies the *parallelogram law*

$$\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2,$$

which fails for $\|\cdot\|_p$ when $p \ne 2$. The inner-product structure makes L^2 uniquely well-suited for geometric questions: angles, orthogonality, projections.

A complete normed space whose norm comes from an inner product is called a *Hilbert space*. So $L^2[a,b]$ is a Hilbert space.

Orthogonality.

Definition 2.18. Two functions $f, g \in L^2[a,b]$ are *orthogonal*, written $f \perp g$, if $\langle f, g\rangle = 0$. A family of functions $\{f_n\}$ is called an *orthogonal family* if $\langle f_n, f_m\rangle = 0$ for all $n \ne m$.

Example 2.19. The family $\{\sin(nx) : n = 1, 2, 3, \dots\}$ is orthogonal in $L^2[0,\pi]$. We compute, for $m \ne n$:

$$\int_0^\pi \sin(mx)\sin(nx)\,dx = \frac{1}{2}\int_0^\pi \big[\cos((m-n)x) - \cos((m+n)x)\big]\,dx,$$

using the product-to-sum identity. Since $m + n \ne 0$ and $m - n \ne 0$ (as $m \ne n$), both cosines integrate to zero over $[0,\pi]$:

$$\int_0^\pi \cos(kx)\,dx = \frac{\sin(k\pi)}{k} = 0 \qquad \text{for any nonzero integer } k.$$

Therefore $\langle \sin(mx), \sin(nx)\rangle = 0$ whenever $m \ne n$.

Remark. The orthogonality of $\{\sin(nx)\}$ on $[0,\pi]$ is not a coincidence. These functions are the eigenfunctions of the differential operator $-d^2/dx^2$ with Dirichlet boundary conditions $y(0) = y(\pi) = 0$, with eigenvalues $\lambda_n = n^2$. The general fact, which we will establish in the Sturm–Liouville theory of the Prüfer track, is that the eigenfunctions of a self-adjoint operator corresponding to distinct eigenvalues are orthogonal.

3. Linear Operators on Function Spaces

In Dr. Nance's Tuesday lecture, we studied linear operators on general vector spaces. The same notion applies to function spaces, with the operators acting on functions to produce new functions. In this section we develop the basic theory: kernel and image, continuity, boundedness, the operator norm, and self-adjointness.

Linear Operators.

Definition 3.1. Let V and W be function spaces. A map $T : V \to W$ is a *linear operator* if, for all $f, g \in V$ and all $\alpha \in \mathbb{R}$,

$$T(f+g) = T(f) + T(g) \qquad \text{and} \qquad T(\alpha f) = \alpha\, T(f).$$

Example 3.2 (Differentiation)**.** The *differentiation operator* $D : C^1[a,b] \to C[a,b]$ defined by $D(f) := f'$ is linear:

$$D(f+g) = (f+g)' = f' + g' = D(f) + D(g), \qquad D(\alpha f) = (\alpha f)' = \alpha f' = \alpha D(f).$$

Example 3.3 (Integration)**.** For a fixed point $c \in [a,b]$, the operator $I : C[a,b] \to C^1[a,b]$ defined by

$$I(f)(x) := \int_c^x f(t)\, dt$$

is linear, by linearity of the integral.

Example 3.4 (Multiplication)**.** Let $\phi \in C[a,b]$ be a fixed function. The *multiplication operator* $M_\phi : C[a,b] \to C[a,b]$ defined by $M_\phi(f) := \phi \cdot f$ (pointwise product) is linear.

Example 3.5 (Point Evaluation)**.** For a fixed $c \in [a,b]$, the map $\mathrm{ev}_c : C[a,b] \to \mathbb{R}$ defined by $\mathrm{ev}_c(f) := f(c)$ is a linear operator. Operators with output in $\mathbb{R}$ are called *linear functionals*.

Example 3.6 (Integral Functional)**.** The map $J : C[a,b] \to \mathbb{R}$ defined by $J(f) := \int_a^b f(x)\, dx$ is a linear functional.

Kernel and Image. Every linear operator has two associated function spaces that capture its essential behavior.

Definition 3.7. Let $T : V \to W$ be a linear operator. The *kernel* of T is

$$\ker(T) := \{f \in V : T(f) = 0\},$$

and the *image* of T is

$$\mathrm{im}(T) := \{T(f) : f \in V\} \subseteq W.$$

Proposition 3.8. *Let $T : V \to W$ be a linear operator. Then $\ker(T)$ is a function space on the domain of V, and $\mathrm{im}(T)$ is a function space on the domain of W.*

Proof. We verify the three conditions of the Subspace Test for each.

Kernel. The zero function $0_V \in V$ satisfies $T(0_V) = 0_W$ by linearity (taking $\alpha = 0$ in the homogeneity axiom), so $0_V \in \ker(T)$. If $f, g \in \ker(T)$, then $T(f+g) = T(f) + T(g) = 0 + 0 = 0$, so $f + g \in \ker(T)$. If $f \in \ker(T)$ and $\alpha \in \mathbb{R}$, then $T(\alpha f) = \alpha T(f) = \alpha \cdot 0 = 0$, so $\alpha f \in \ker(T)$.

Image. The zero function 0_W equals $T(0_V)$, so $0_W \in \mathrm{im}(T)$. If $w_1, w_2 \in \mathrm{im}(T)$, write $w_1 = T(f_1)$ and $w_2 = T(f_2)$; then $w_1 + w_2 = T(f_1) + T(f_2) = T(f_1 + f_2)$, which is in $\mathrm{im}(T)$. If $w = T(f) \in \mathrm{im}(T)$ and $\alpha \in \mathbb{R}$, then $\alpha w = \alpha T(f) = T(\alpha f) \in \mathrm{im}(T)$. □

Example 3.9. For the differentiation operator $D : C^1[a,b] \to C[a,b]$,

$$\ker(D) = \{f \in C^1[a,b] : f' \equiv 0\} = \{\text{constant functions on } [a,b]\}.$$

This is a one-dimensional function space.

Example 3.10. For point evaluation $\mathrm{ev}_c : C[a,b] \to \mathbb{R}$,

$$\ker(\mathrm{ev}_c) = \{f \in C[a,b] : f(c) = 0\}.$$