# NONLINEAR BOUNDARY VALUE PROBLEMS

SHALMALI BANDYOPADHYAY

## 1. Initial value problems versus boundary value problems

A second-order differential equation on an interval needs two side conditions to determine a solution, and everything depends on where those conditions are placed.

**Definition 1.1.** Let $[a, b]$ be a closed bounded interval and consider a second-order equation $u'' = g(x, u, u')$ on $(a, b)$. An *initial value problem* imposes both side conditions at the single point $x = a$, as in $u(a) = u_0$ and $u'(a) = m$. A *boundary value problem* splits the conditions between the two endpoints, one at $x = a$ and one at $x = b$.

The decisive fact about initial value problems is that, under a mild condition on the right-hand side, they have one and only one solution.

**Theorem 1.2** (Picard existence and uniqueness)**.** *Let $I \subseteq \mathbb{R}$ be an interval, and let $F : I \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and globally Lipschitz in its second argument: there is a constant $L \geq 0$ with*

$$\|F(t, y) - F(t, z)\| \leq L\, \|y - z\| \qquad \text{for all } t \in I,\ y, z \in \mathbb{R}^n.$$

*Then for each $t_0 \in I$ and each $y_0 \in \mathbb{R}^n$ the initial value problem*

$$y' = F(t, y), \qquad y(t_0) = y_0$$

*has a unique solution, and that solution is defined on all of $I$.*

*Remark.* This is the contrast that drives everything below. An initial value problem fixes both data at one point, and Theorem 1.2 makes its solution exist and be unique. A boundary value problem splits the data between the two ends, and now nothing is automatic: a solution may fail to exist, may be unique, or may be far from unique. The example in Section 2 makes this vivid.

**Definition 1.3.** For a second-order equation on $[a, b]$ the standard boundary conditions are the following.

(i) *Dirichlet* (first kind): $u(a) = \alpha$, $u(b) = \beta$; the value is prescribed, as when a temperature is held fixed at each end.
(ii) *Neumann* (second kind): $u'(a) = \alpha$, $u'(b) = \beta$; the slope, hence the flux, is prescribed, and $u' = 0$ is a no-flux boundary.
(iii) *Robin* (third kind): $\alpha_1 u(a) + \alpha_2 u'(a) = \gamma$ and an analogous relation at $b$; this models convective exchange with the surroundings.
(iv) *Mixed-type*: a condition of one kind at $a$ and of another at $b$, such as $u(a) = \alpha$ with $u'(b) = \beta$.
(v) *Periodic*: $u(a) = u(b)$ and $u'(a) = u'(b)$.

Conditions (i)–(iv) are *separated*, each constraining one endpoint, while (v) couples the two ends. The example below is Dirichlet.

*Date*: UTM NREUP 2026.

## 2. A simple example: the shooting method

Consider the Dirichlet problem

$$u'' + |u| = 0, \qquad u(0) = 0, \quad u(b) = B, \qquad b > 0,\ B \in \mathbb{R}. \tag{$*$}$$

A solution is a curve that leaves height 0 at time 0 and arrives at height $B$ at time $b$ (Figure 1). Three questions present themselves, and they are exactly the questions the remark above warned about: does a solution exist, is it unique, and can there be several?

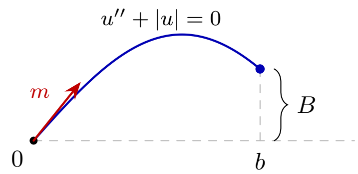



Figure 1. The boundary value problem ($*$) as a shooting problem: launch a curve from the origin and ask whether it can be made to reach the height $B$ at time $b$.

The shooting method answers all three by comparison with an initial value problem. Instead of fixing the height at $t = b$, fix the slope at $t = 0$:

$$u'' + |u| = 0, \qquad u(0) = 0, \quad u'(0) = m. \tag{$**$}$$

For each slope $m \in \mathbb{R}$ the problem ($**$) has a unique solution $u(t, m)$ on $[0, \infty)$, by Lemma 2.1 below. As $m$ ranges over $\mathbb{R}$ these solutions sweep out a whole family of trajectories leaving the origin, one for each slope (Figure 2). Solving the boundary value problem ($*$) is then precisely the problem of selecting, from this family, a trajectory that passes through the target:

$$\text{find } m \in \mathbb{R} \text{ such that } u(b, m) = B.$$

Here is the contrast in a sentence: the initial value problem ($**$) always has exactly one solution, whereas whether ($*$) has none, one, or many solutions depends entirely on whether and how often the family in Figure 2 meets the target line.

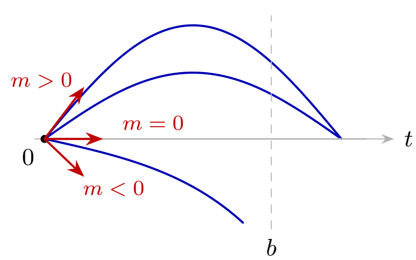



Figure 2. The initial value problem ($**$) has one trajectory for each slope $m$. Boundary value problem ($*$) asks which of these trajectories, if any, lands on the height $B$ at $t = b$.

**The trajectories.** Recall $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$.

**Lemma 2.1** (Trajectories)**.** *For every $m \in \mathbb{R}$ the initial value problem ($**$) has a unique solution on $[0, \infty)$, given by*

$$u(t, m) = \begin{cases} m \sin t, & 0 \le t \le \pi, \\ -m \sinh(t - \pi), & t \ge \pi, \end{cases} \quad (m > 0), \qquad u(t, 0) \equiv 0, \qquad u(t, m) = m \sinh t \ \ (m < 0).$$

*Proof. Existence and uniqueness.* Write $(**)$ as a first-order system for the state $Y=(u,u')$, namely $Y'=F(Y)$ with $F(u,v)=(v,-|u|)$. For any two states,

$$\|F(u_1,v_1)-F(u_2,v_2)\| = \big\|\big(v_1-v_2,\ -(|u_1|-|u_2|)\big)\big\| \le \|(u_1-u_2,\ v_1-v_2)\|\,,$$

since $\big||u_1|-|u_2|\big| \le |u_1-u_2|$. Thus $F$ is globally Lipschitz with constant 1, and Theorem 1.2 gives a unique solution on all of $[0,\infty)$ for each slope $m$. It remains to exhibit it; by uniqueness, any function we construct that solves $(**)$ must be that solution.

*Construction.* Suppose $m>0$. While $u>0$ we have $|u|=u$, so the equation is $u''=-u$, with general solution $a\cos t+b\sin t$; the conditions $u(0)=0$ and $u'(0)=m$ force $a=0$ and $b=m$, so $u=m\sin t$. This is positive on $(0,\pi)$ and returns to 0 at $t=\pi$, where $u'(\pi)=m\cos\pi=-m<0$. For $t\ge\pi$ the curve is below the axis, so $|u|=-u$ and the equation is $u''=u$, with general solution $c\cosh(t-\pi)+d\sinh(t-\pi)$; matching the value $u(\pi)=0$ gives $c=0$, and matching the slope $u'(\pi)=-m$ gives $d=-m$, so $u=-m\sinh(t-\pi)$, which is negative for all $t>\pi$. The two pieces share the value 0 and the slope $-m$ at $t=\pi$, and there $u''=0$ on both sides, so the patched function is $C^2$ and solves $(**)$ on $[0,\infty)$.

If $m=0$, then $u\equiv 0$ solves $(**)$. If $m<0$, then $u<0$ for small $t>0$, so $|u|=-u$ and $u''=u$; with $u(0)=0$ and $u'(0)=m$ this gives $u=m\sinh t$, which stays negative for all $t>0$. In each case the displayed formula is, by the first part, the unique solution. □

**From trajectories to a count.** By Lemma 2.1, solving $(*)$ means finding a slope $m$ with $u(b,m)=B$. Define the *shooting map*

$$\varphi_b(m) := u(b,m),$$

the height of the trajectory at the far endpoint $t=b$. A solution of $(*)$ is exactly a slope $m$ with $\varphi_b(m)=B$, so

$$\#\{\text{solutions of } (*)\} = \#\{\, m\in\mathbb{R} : \varphi_b(m)=B\,\},$$

the number of times the horizontal line at height $B$ meets the graph of $\varphi_b$. Evaluating Lemma 2.1 at $t=b$ gives

$$\varphi_b(m) = \begin{cases} m\sin b\ (m\ge 0), \quad m\sinh b\ (m\le 0), & b<\pi,\\ 0\ (m\ge 0), \quad m\sinh\pi\ (m\le 0), & b=\pi,\\ -m\sinh(b-\pi)\ (m\ge 0), \quad m\sinh b\ (m\le 0), & b>\pi, \end{cases}$$

where the sign in the $b>\pi$ row comes from the second piece of the $m>0$ formula in Lemma 2.1, since at $t=b>\pi$ a positive launch has already crossed below the axis. The three regimes give three different graphs of $\varphi_b$, drawn in Figure 3, and the classification below is read directly off them.

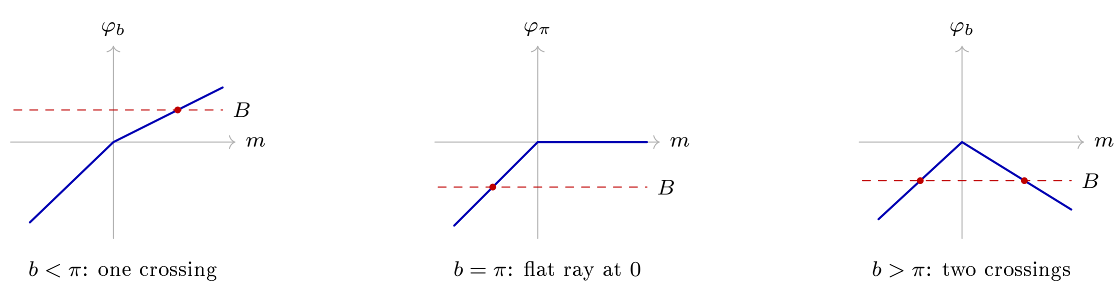



FIGURE 3. The shooting map $\varphi_b$ as a function of the launch slope $m$. The number of solutions of $(*)$ for a given $B$ is the number of times the horizontal line at height $B$ meets the graph: an increasing line through the origin when $b<\pi$, a ray flat at 0 joined to a downward line when $b=\pi$, and an inverted "$\Lambda$" with peak 0 when $b>\pi$.

**Theorem 2.2** (Classification). *Let $b > 0$ and $B \in \mathbb{R}$. The boundary value problem $(*)$ has*

(i) *a unique solution for every $B$, when $b < \pi$;*
(ii) *no solution if $B > 0$, infinitely many if $B = 0$, and a unique solution if $B < 0$, when $b = \pi$;*
(iii) *no solution if $B > 0$, the unique solution $u \equiv 0$ if $B = 0$, and exactly two solutions if $B < 0$, when $b > \pi$.*

*The eight nontrivial configurations are drawn in Figure 4, and the verdict across the $(b, B)$ plane is shown in Figure 5.*

*Proof.* We count the solutions $m$ of $\varphi_b(m) = B$ in each regime; see Figure 3.

*Case $b < \pi$.* Both pieces of $\varphi_b$ have positive slope, since $\sin b > 0$ and $\sinh b > 0$, and they meet at $\varphi_b(0) = 0$. Hence $\varphi_b$ is continuous and strictly increasing on $\mathbb{R}$, with $\varphi_b(m) \to \pm\infty$ as $m \to \pm\infty$; it is therefore a bijection $\mathbb{R} \to \mathbb{R}$, so $\varphi_b(m) = B$ has exactly one solution for every $B$. Explicitly, $u(t) = \frac{B}{\sin b} \sin t$ when $B > 0$, $u \equiv 0$ when $B = 0$, and $u(t) = \frac{B}{\sinh b} \sinh t$ when $B < 0$.

*Case $b = \pi$.* Now $\varphi_\pi(m) = 0$ for every $m \geq 0$, while $\varphi_\pi(m) = m \sinh \pi$ for $m \leq 0$, so the range of $\varphi_\pi$ is $(-\infty, 0]$. If $B > 0$ there is no preimage, hence no solution. If $B = 0$ the preimage is the entire ray $m \geq 0$, giving the infinite family $u = m \sin t$. If $B < 0$ the unique preimage is $m = B/\sinh \pi$, so $u(t) = \frac{B}{\sinh \pi} \sinh t$.

*Case $b > \pi$.* Here $\varphi_b$ has slope $-\sinh(b - \pi) < 0$ on $m \geq 0$ and slope $\sinh b > 0$ on $m \leq 0$, with $\varphi_b(0) = 0$. From the peak value 0 at $m = 0$ the graph decreases to $-\infty$ on both sides, so the range is again $(-\infty, 0]$. If $B > 0$ there is no solution; if $B = 0$ the only preimage is $m = 0$, the trivial solution $u \equiv 0$; if $B < 0$ the line at height $B$ meets each branch once, at $m = B/\sinh b < 0$ and at $m = -B/\sinh(b - \pi) > 0$, giving the two solutions

$$u_1(t) = \begin{cases} \dfrac{-B}{\sinh(b - \pi)} \sin t, & t \in [0, \pi], \\ \dfrac{B}{\sinh(b - \pi)} \sinh(t - \pi), & t \in [\pi, b], \end{cases} \qquad u_2(t) = \frac{B}{\sinh b} \sinh t.$$

□

*Remark.* The wall $b = \pi$ is an eigenvalue statement in disguise. The small-amplitude behavior of $(*)$ is governed by the linearization $u'' + u = 0$, and $\pi$ is exactly the shortest interval on which that equation has a nontrivial solution vanishing at both ends, namely $\sin t$; equivalently $\lambda = 1$ is the first Dirichlet eigenvalue of $-u'' = \lambda u$ on $(0, \pi)$. The bookkeeping of zero crossings that drives the proof is what oscillation theory, and the Prüfer transformation in particular, is built to track.

## 3. The quadrature method

We now consider a parametrized family of nonlinear Dirichlet problems on $(0, 1)$ in which the equation is *autonomous*, carrying no explicit dependence on the independent variable. Throughout, $\lambda > 0$ is a constant parameter and $f \in C^1$ satisfies $f > 0$. We seek *positive* solutions of

$$-u'' = \lambda f(u) \quad \text{on } (0, 1), \qquad u(0) = 0 = u(1), \qquad u > 0. \tag{1}$$

The autonomy is decisive: it forces every positive solution to be symmetric, and that symmetry converts the differential equation into a single integral, the *quadrature*.

**Lemma 3.1** (Symmetry). *If $u$ solves the equation in (1) and $u'(x_0) = 0$ for some interior $x_0$, then $u(x_0 - x) = u(x_0 + x)$ for all admissible $x$.*

*Proof.* Set $v(x) = u(x_0 + x)$ and $w(x) = u(x_0 - x)$. Then $v''(x) = u''(x_0 + x) = -\lambda f(u(x_0 + x)) = -\lambda f(v)$, and likewise $w''(x) = u''(x_0 - x) = -\lambda f(w)$, so both satisfy $-z'' = \lambda f(z)$. Moreover $v(0) = w(0) = u(x_0)$, and $v'(0) = u'(x_0) = 0$ while $w'(0) = -u'(x_0) = 0$. Thus $v$ and $w$ solve the same initial value problem, and by uniqueness $v \equiv w$. □

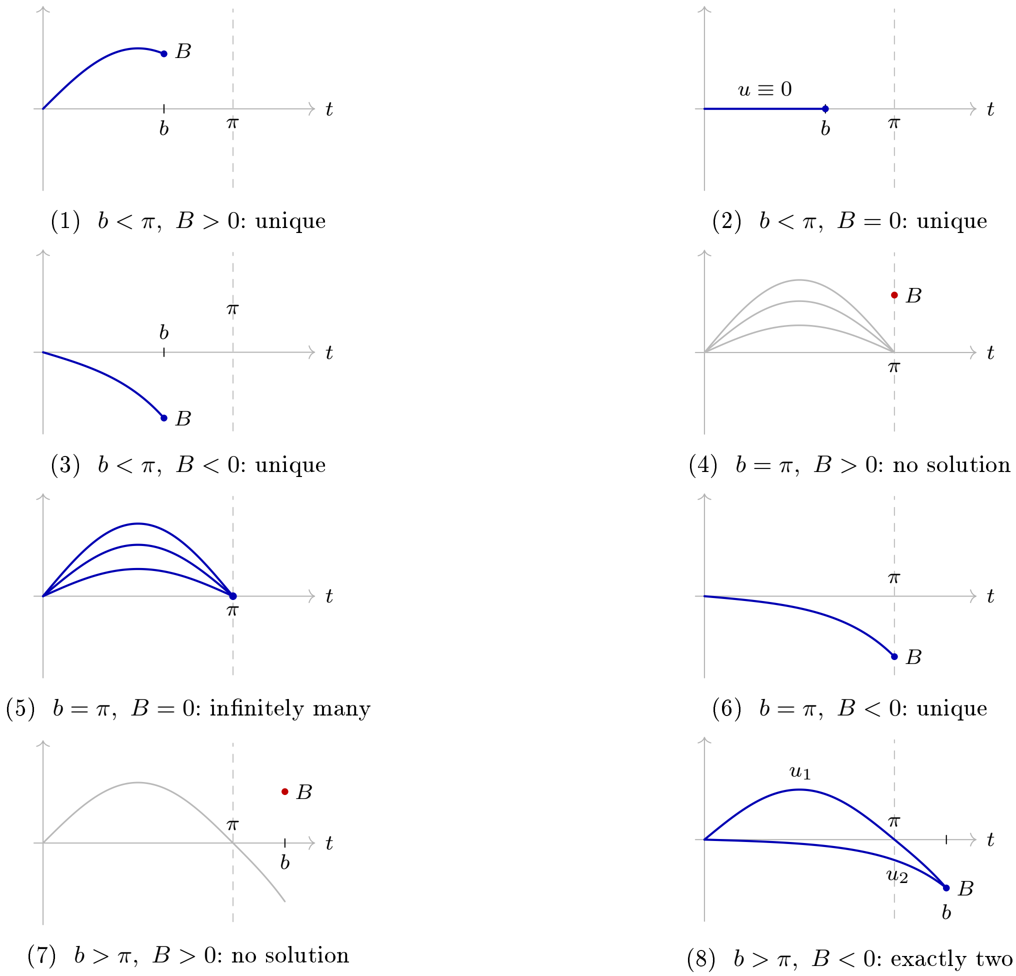



FIGURE 4. Shooting trajectories for $(*)$. Blue curves are solutions, gray curves are failed shooting attempts, and a red dot marks an unreachable target $(b, B)$. The dashed line is the critical length $t = \pi$.

**Corollary 3.2.** *A positive solution of* (1) *is symmetric about the midpoint* $x = \frac{1}{2}$*, where it attains its maximum* $\rho := u(\frac{1}{2}) = \|u\|_\infty$.

*Proof.* A positive solution vanishes at both ends and is positive inside, so it has an interior maximum, where $u' = 0$. By Lemma 3.1 it is symmetric about that point; since the two boundary zeros are reflections of one another, the point of symmetry is the midpoint. □

**From the equation to a quadrature.** Let $F(s) := \int_0^s f(t)\,dt$, so that $F' = f$ and $F(0) = 0$. Multiply the equation $-u'' = \lambda f(u)$ by $u'$:

$$-u''u' = \lambda f(u)\,u'.$$

Both sides are exact derivatives, since $\frac{d}{dx}\big[\frac{1}{2}(u')^2\big] = u''u'$ and $\frac{d}{dx}\big[F(u)\big] = f(u)u'$. Hence

$$\frac{d}{dx}\Big[\tfrac{1}{2}(u')^2 + \lambda F(u)\Big] = 0,$$

so the *energy* $E := \frac{1}{2}(u')^2 + \lambda F(u)$ is constant along a solution. Evaluating $E$ at the midpoint, where $u' = 0$ and $u = \rho$ by Corollary 3.2, gives $E = \lambda F(\rho)$, and therefore

$$(u')^2 = 2\lambda\big(F(\rho) - F(u)\big) \qquad \text{on } [0,1]. \tag{2}$$

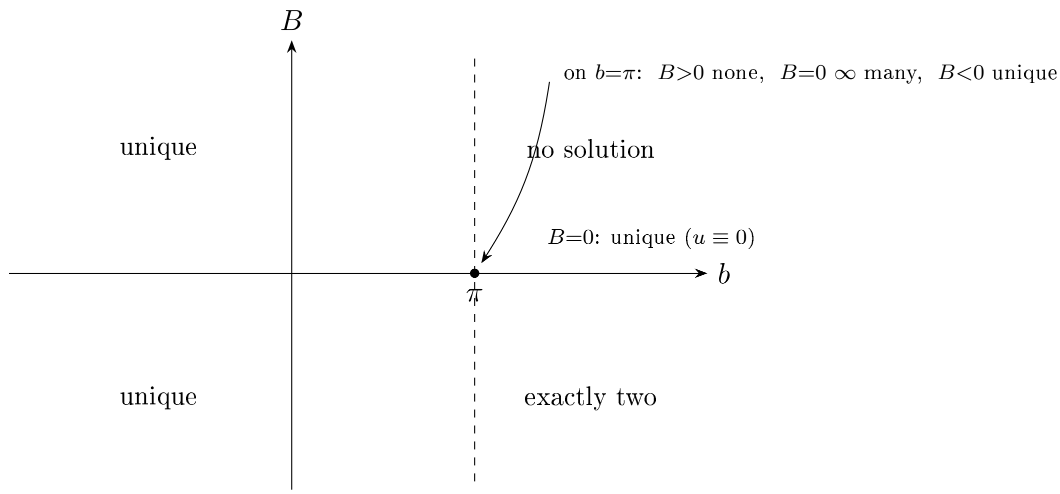



FIGURE 5. Solution count of (∗) as a function of the interval length $b$ and the boundary value $B$.

On the left half $[0, \frac{1}{2}]$ the solution rises from 0 to $\rho$, so $u' > 0$ there, and (2) gives $u' = \sqrt{2\lambda}\,\sqrt{F(\rho) - F(u)}$. Separating variables and using $u(0) = 0$,

$$\int_0^{u(x)} \frac{\mathrm{d}z}{\sqrt{F(\rho) - F(z)}} = \sqrt{2\lambda}\,x, \qquad x \in [0, \tfrac{1}{2}]. \tag{3}$$

This is the quadrature: it determines $u$ on $[0, \frac{1}{2}]$ by a single integral, and by symmetry on all of $[0, 1]$.

**Theorem 3.3** (Quadrature characterization; Laetsch, 1970)**.** *For $\rho > 0$ set*

$$G(\rho) := \sqrt{2}\int_0^{\rho} \frac{\mathrm{d}z}{\sqrt{F(\rho) - F(z)}}.$$

*A positive function $u$ with $\|u\|_\infty = \rho$ solves* (1) *if and only if*

$$\sqrt{\lambda} = G(\rho).$$

*When this holds, $u$ is given on $[0, \frac{1}{2}]$ by* (3) *and on $[\frac{1}{2}, 1]$ by reflection about the midpoint.*

*Proof.* ($\Rightarrow$) Let $u$ solve (1) with $\|u\|_\infty = \rho$. By Corollary 3.2 its maximum $\rho$ is attained at $x = \frac{1}{2}$, and the derivation above produces (3) on $[0, \frac{1}{2}]$. Let $x \to \frac{1}{2}^-$. The right side tends to $\sqrt{2\lambda}\cdot\frac{1}{2} = \sqrt{\lambda/2}$, while $u(x) \to \rho$, so the left side tends to $\int_0^{\rho} \frac{\mathrm{d}z}{\sqrt{F(\rho)-F(z)}}$, which is finite by Proposition 3.4 below. Hence

$$\int_0^{\rho} \frac{\mathrm{d}z}{\sqrt{F(\rho) - F(z)}} = \sqrt{\tfrac{\lambda}{2}},$$

and multiplying by $\sqrt{2}$ gives $G(\rho) = \sqrt{\lambda}$.

($\Leftarrow$) Conversely, suppose $\sqrt{\lambda} = G(\rho)$ for some $\rho > 0$. Define

$$H(s) = \int_0^{s} \frac{\mathrm{d}z}{\sqrt{F(\rho) - F(z)}}, \qquad 0 \le s \le \rho,$$

which is continuous and strictly increasing, with $H(0) = 0$ and $H(\rho) = G(\rho)/\sqrt{2} = \sqrt{\lambda}/\sqrt{2} = \sqrt{\lambda/2}$. Define $u$ on $[0, \frac{1}{2}]$ implicitly by $H(u(x)) = \sqrt{2\lambda}\,x$. At $x = 0$ this gives $u(0) = 0$, and at $x = \frac{1}{2}$ it gives

$H(u(\frac{1}{2})) = \sqrt{2\lambda}\cdot\frac{1}{2} = \sqrt{\lambda/2} = H(\rho)$, so $u(\frac{1}{2}) = \rho$ and the definition is consistent. Differentiating $H(u) = \sqrt{2\lambda}\,x$ with $H'(s) = 1/\sqrt{F(\rho) - F(s)}$ yields

$$\frac{u'(x)}{\sqrt{F(\rho) - F(u)}} = \sqrt{2\lambda}, \quad \text{so} \quad (u')^2 = 2\lambda\big(F(\rho) - F(u)\big),$$

and in particular $u'(\frac{1}{2}) = 0$ since $u(\frac{1}{2}) = \rho$. Differentiating $(u')^2 = 2\lambda(F(\rho) - F(u))$ gives $2u'u'' = -2\lambda f(u)u'$, hence $-u'' = \lambda f(u)$ wherever $u' \neq 0$, and by continuity everywhere. Extending $u$ to $[\frac{1}{2}, 1]$ by reflection about $\frac{1}{2}$ produces a positive solution of (1) with $\|u\|_\infty = \rho$. □

*Remark.* The theorem says that, for a fixed $\lambda$, the positive solutions of (1) correspond exactly to the heights $\rho > 0$ with $G(\rho) = \sqrt{\lambda}$. The graph of $G$ is therefore the bifurcation diagram: a horizontal line at height $\sqrt{\lambda}$ meets it once for each solution, so the three questions of Section 2 are answered here by counting intersections.

The function $G$ is an improper integral whose integrand blows up at the upper limit, so it must be shown finite.

**Proposition 3.4.** *$G(\rho)$ is finite for every $\rho > 0$.*

*Proof.* The only difficulty is at $z = \rho$, where $F(\rho) - F(z) = 0$. Since $f$ is continuous and $f(\rho) > 0$, there are $\varepsilon \in (0, \rho)$ and $m > 0$ with $f \geq m$ on $[\rho - \varepsilon, \rho]$. For such $z$,

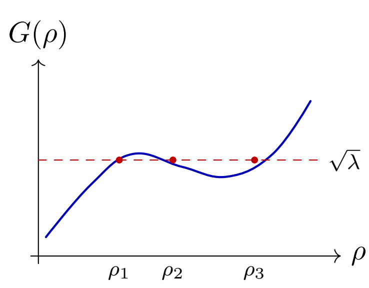
$$F(\rho) - F(z) = \int_z^\rho f(s)\,\mathrm{d}s \;\geq\; m(\rho - z),$$

so the integrand is at most $\big(m(\rho - z)\big)^{-1/2}$ and

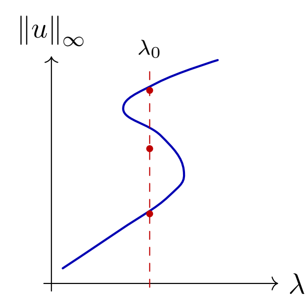
$$\int_{\rho-\varepsilon}^{\rho} \frac{\mathrm{d}z}{\sqrt{m(\rho - z)}} = \frac{2}{\sqrt{m}}\sqrt{\varepsilon} < \infty.$$

On $[0, \rho - \varepsilon]$ the integrand is continuous and bounded, so its contribution is finite too. Hence $G(\rho) < \infty$. □

**Example 3.5** (Combustion)**.** Take $f(s) = \exp\big(\alpha s/(\alpha + s)\big)$ with $\alpha > 0$, a positive, increasing, bounded reaction term. The shape of $G$ depends sharply on $\alpha$. For small $\alpha$, $G$ is increasing and the line $\sqrt{\lambda}$ meets it once, so (1) has a unique positive solution growing with $\lambda$. For large $\alpha$, $G$ folds, rising, then falling, then rising again, so over a range of $\lambda$ the line meets it three times and there are three positive solutions. Plotting $\|u\|_\infty$ against $\lambda$ gives the S-shaped ignition curve of Figure 6, whose two turning points mark ignition and extinction.




the time map $G$ for $\alpha \gg 1$




the resulting S-shaped branch

FIGURE 6. For large $\alpha$ the map $G$ folds, so one value of $\lambda$ can yield three positive solutions; the branch of maxima is the S-shaped combustion curve.

## 4. Exercises

These problems accompany the notes above. Throughout, $(*)$ denotes the Dirichlet problem

$$u'' + |u| = 0, \qquad u(0) = 0, \quad u(b) = B, \qquad b > 0,\ B \in \mathbb{R},$$

and (Q) denotes the autonomous problem

$$-u'' = \lambda f(u) \ \text{ on } (0,1), \qquad u(0) = 0 = u(1), \qquad u > 0,$$

with $\lambda > 0$, $f \in C^1$, and $f > 0$. Recall $F(s) = \int_0^s f(t)\,\mathrm{d}t$ and the time map $G(\rho) = \sqrt{2}\int_0^\rho \frac{\mathrm{d}z}{\sqrt{F(\rho)-F(z)}}$.

**Exercise 1.** For $0 < b < \pi$ and $B > 0$, solve $(*)$ directly, without shooting, by solving $u'' + u = 0$ on $[0,b]$ with $u(0) = 0$ and $u(b) = B$. Confirm that $u(t) = \dfrac{B}{\sin b}\sin t$, and explain what goes wrong as $b \to \pi^-$.

**Exercise 2.** Take $b > \pi$ and $B < 0$, the case in which $(*)$ has exactly two solutions. Verify that both

$$u_1(t) = \begin{cases} \dfrac{-B}{\sinh(b-\pi)}\sin t, & t \in [0,\pi], \\ \dfrac{B}{\sinh(b-\pi)}\sinh(t-\pi), & t \in [\pi, b], \end{cases} \qquad u_2(t) = \frac{B}{\sinh b}\sinh t$$

satisfy $u(0) = 0$ and $u(b) = B$, and check that $u_1$ is continuously differentiable across $t = \pi$.

**Exercise 3.** Carry out the quadrature for $f(u) = e^u$ in (Q). Write $F$ and $G(\rho)$ explicitly and sketch the graph of $G$. For which values of $\lambda$ does (Q) have a positive solution, and how many?

**Exercise 4.** For $f(u) = u^p$ with $p > 1$ in (Q), show that $G(\rho)$ scales as a power of $\rho$; find the exponent. Use this to describe the number of positive solutions as a function of $\lambda$.

**Exercise 5.** The symmetry lemma for (Q) states: if $u$ solves $-u'' = \lambda f(u)$ and $u'(x_0) = 0$ at an interior point $x_0$, then $u$ is symmetric about $x_0$. Notice that its proof uses only the autonomous equation, not the boundary conditions. Restate and reprove the lemma for the Neumann problem $-u'' = \lambda f(u)$ on $(0,1)$ with $u'(0) = 0 = u'(1)$, use it to locate the critical points of a solution, and determine where the maximum sits. How does this differ from the Dirichlet case?