Boundary Value Problems: Guided Discovery

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Work through the phases in order, in your group. Each phase sets up the next; reach the result yourselves before moving on.

Discovery A

Build, by hand, every solution of

$$u'' + |u| = 0, \qquad u(0) = 0, \quad u(b) = B \qquad (b > 0,\ B \in \mathbb{R}),$$

and watch the number of solutions change with the interval length b.

Phase 1. While $u > 0$, the absolute value disappears and the equation is linear: $u'' + u = 0$. Write its general solution and impose $u(0) = 0$. What single shape does this leave you?

Phase 2. Suppose $0 < b < \pi$ and $B > 0$. Use $u(b) = B$ to pin down the remaining constant, and write the solution. Now imagine increasing b toward π: what happens to your formula, and why does the construction break down exactly at $b = \pi$?

Phase 3. Take $b > \pi$ and $B < 0$. A curve can no longer stay positive all the way across. Build a solution in two pieces: a positive arch on $[0, \pi]$ where $u'' + u = 0$, joined to a piece on $[\pi, b]$ where $u < 0$ and so $u'' - u = 0$. Require the two pieces to agree in *both* value and slope at $t = \pi$, then impose $u(b) = B$. Solve for all constants.

Phase 4. For the same $b > \pi$ and $B < 0$, find a *different* solution that stays below the axis the whole way (so $u'' - u = 0$ on all of $[0, b]$). Counting this together with Phase 3, how many solutions are there now?

Phase 5. *Conclusion.* Record the number of solutions when $0 < b < \pi$, when $b = \pi$, and when $b > \pi$. In one sentence, say what makes $b = \pi$ the dividing length.

Discovery B

For $-u'' = \lambda f(u)$ *on* $(0,1)$, $u(0) = 0 = u(1)$, $u > 0$ *(with* $\lambda > 0$, $f > 0$*), a positive solution is symmetric about* $x = \frac{1}{2}$ *with height* $\rho = \|u\|_\infty$*, and the energy identity reduces the problem to the time map*

$$\sqrt{\lambda} = G(\rho) := \sqrt{2}\int_0^\rho \frac{\mathrm{d}z}{\sqrt{F(\rho) - F(z)}}, \qquad F(s) = \int_0^s f(t)\,\mathrm{d}t.$$

Discover how the shape of G *decides the number of solutions.*

Phase 1. Take $f(u) = u^p$ with $p > 1$. Compute $F(s)$. In the integral for $G(\rho)$, substitute $z = \rho w$ to pull every factor of ρ outside the integral. Show that

$$G(\rho) = C\,\rho^{(1-p)/2},$$

and write the constant C as a (finite) integral over $[0,1]$.

Phase 2. Is the exponent $(1-p)/2$ positive or negative? Sketch $G(\rho)$ accordingly, then draw a horizontal line at height $\sqrt{\lambda}$. For a given $\lambda > 0$, how many times does the line meet the curve, and therefore how many positive solutions does the problem have?

Phase 3. Now take $f(u) = e^u$. Write $F(s)$ and the time map $G(\rho)$ (simplify $F(\rho) - F(z)$). Estimate G at a few values of ρ and sketch it. How does its shape differ from the power case, and what does that say about the number of positive solutions as λ increases from 0?

Phase 4. *Conclusion.* State, in one sentence, the rule that turns the graph of G into a count of positive solutions for each λ.