Prüfer Transformation of Discrete Non-Uniform Sturm-Liouville Problem

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The Discrete Non-uniform Sturm-Liouville Problem

The general Discrete Non-Uniform Sturm–Liouville (SL) equation:

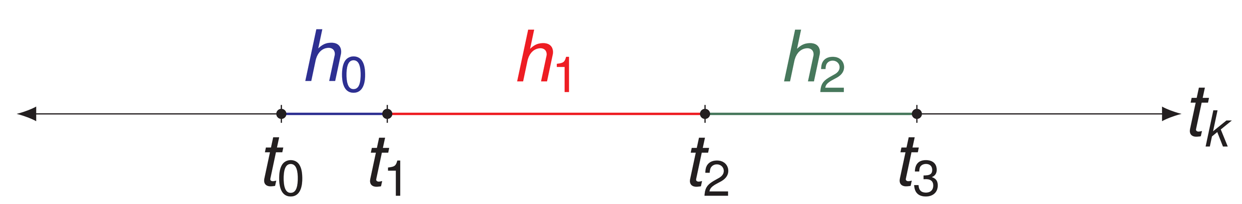
$$\Delta_h (r_k \Delta_h x_k) + (\lambda w_k - q_k)\, x_{k+1} = 0 \qquad \text{for } k = 0, \ldots, n-1$$

with a Dirichlet boundary:

$$x_0 = x_{n+1} = 0$$

where:

- $\Delta_h x_k = \dfrac{x_{k+1} - x_k}{t_{k+1} - t_k} = \dfrac{x_{k+1} - x_k}{h_k}$
- $r_k, w_k, q_k : \mathbb{N} \to \mathbb{R}$ with $r_k \neq 0$ & $w_k > 0$ are weight functions.
- $\lambda \in \mathbb{R}$ is a parameter $\to$ EVP/Spectral Problem



Why Prüfer Substitution?

Prüfer Subsitution: Cartesian $\to$ Polar

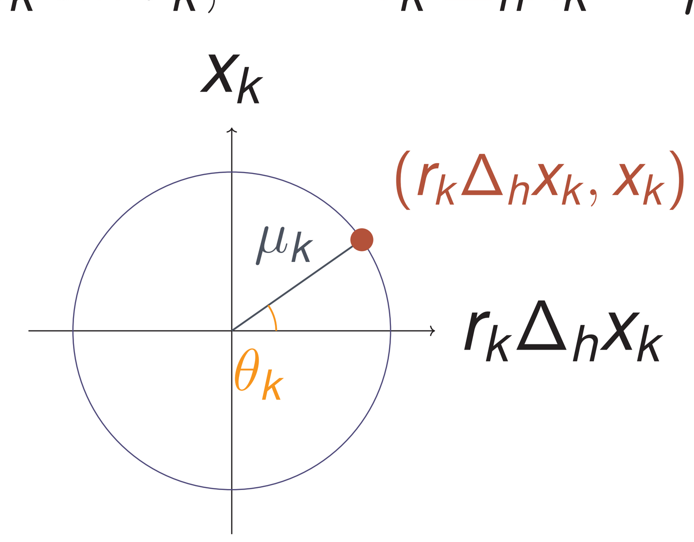
SL-BVP $\overset{\text{Prüfer Substitution}}{\Longleftrightarrow}$ 1$^{\text{st}}$ order D.E. $\overset{\text{Impose Initial Condition}}{\Longleftrightarrow}$ system of IVP

(amplitude + phase)

The Prüfer Substitution

Switch to polar coordinates in the $(r_k \Delta_h x_k, x_k)$ plane:

$$x_k = \mu_k \sin\theta_k, \qquad r_k \Delta_h x_k = \mu_k \cos\theta_k$$



References

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Applying Discrete Derivative to Substitution

Apply Δ_h to $x_k = \mu_k \sin\theta_k$ and use $\Delta_h x_k = \dfrac{\mu_k \cos\theta_k}{r_k}$:

Equation A:

$$\frac{\mu_k \cos\theta_k}{r_k} = \sin\theta_{k+1}\Delta_h \mu_k + \mu_k \Delta_h \sin\theta_k$$

Apply Δ_h to $r_k \Delta_h x_k = \mu_k \cos\theta_k$ and substitute the SL equation:

Equation B:

$$-\frac{(\lambda w_k - q_k)\; h_k\, \mu_k}{r_k}\cos\theta_k - (\lambda w_k - q_k)\; \mu_k \sin\theta_k = \cos\theta_{k+1}\, \Delta_h \mu_k + \mu_k\, \Delta_h \cos\theta_k$$

Deriving the Prüfer System

Eliminate $\Delta_h \theta_k$: multiply A by $\sin\theta_{k+1}$, B by $\cos\theta_{k+1}$, add and simplify:

$$\boxed{\Delta_h \mu_k = -\mu_k \left(\frac{h_k\left[(\Delta_h \sin\theta_k)^2 + (\Delta_h \cos\theta_k)^2\right]}{2} - \frac{1}{r_k}\sin\theta_{k+1}\cos\theta_k + \frac{(\lambda w_k - q_k)\; h_k}{r_k}\cos\theta_{k+1}\cos\theta_k + (\lambda w_k - q_k)\cos\theta_{k+1}\sin\theta_k \right)}$$

Eliminate $\Delta_h \mu_k$: multiply A by $\cos\theta_{k+1}$, B by $\sin\theta_{k+1}$, subtract and simplify:

$$\boxed{\sin(h_k\, \Delta_h \theta_k) = \frac{h_k}{r_k}\left(\cos\theta_k \cos\theta_{k+1} + (\lambda w_k - q_k)\; h_k \cos\theta_k \sin\theta_{k+1} + (\lambda w_k - q_k)\; r_k \sin\theta_k \sin\theta_{k+1} \right)}$$

Consequences of the Prüfer System

$\Delta_h \theta_k$ is independent of μ_k – solve the θ_k equation alone.

$$x_0 = x_{n+1} = 0 \iff \theta_{n+1}(\lambda) = m\pi \text{ for some } m \in \mathbb{Z}^+$$

Solution to phase equation $\to$ Solution to amplitude equation $\to$ Solution to SL-BVP

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