Numerical Implementation of Sturm-Louiville Problem via Prüfer based Shooting Method

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Discrete Sturm-Liouville Problem via Prüfer Substitution

Sturm-Liouville problem with a Dirichlet boundary condition:

$$\begin{cases} \Delta_h \left(r_k \Delta_h x_k \right) + \left[\lambda w_k - q_k \right] x_{k+1} = 0, & k = 1, \ldots, n-1 \\ x_0 = x_{n+1} = 0 \end{cases}$$

where

- $\Delta_h x_k = \dfrac{x_{k+1} - x_k}{t_{k+1} - t_k} = \dfrac{x_{k+1} - x_k}{h_k}$ is the Non-uniform step size
- $r_k, w_k, q_k : \mathbb{R} \to \mathbb{R}$ which are called **weight functions**, $w_k > 0, r_k \neq 0$
- $\lambda \in \mathbb{R}$ is a parameter.
- $r_k \Delta_h x_k$ is the quasiderivative

Today's Focus: The Test Case

Generalized Problem → Test Problem

- ✓ Take $\Delta_h\left(r_k \Delta_h x_k\right) + \left[\lambda w_k - q_k\right] x_{k+1} = 0$
- ✓ Let $r_k \equiv 1$, $w_k \equiv 1$, $q_k \equiv 0$
- ✓ Then our test case: $\Delta_h\left(\Delta_h x_k\right) + \lambda x_{k+1} = 0$

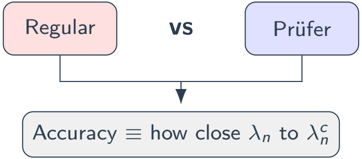
Continuous Version of Test Problem

- $x'' + \lambda_n^c x = 0$ (Linear second-order ODE with homogeneous with constant coefficient.
- Solve using Characteristic equation $\implies \lambda_n^c = n^2$

Today's Goal

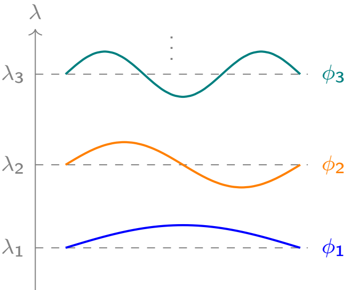
Compare the Two Shooting Methods

- Describe *regular shooting*
- Describe *Prüfer based shooting*
- Compare the algorithm of both methods
- Compare their accuracy on different grids uniform vs non uniform



Oscillatory Behavior of Eigen Functions

- Counting number of zeros of EF's.
- ϕ_n has $n-1$ zeros.



From BVP to IVP: Regular Shooting

The obstacle (Second order BVP)

$$x_0 = 0 \;\; \text{at } a, \qquad x_{n+1} = 0 \;\; \text{at } b.$$

The fix: shoot (Second order IVP)

- Fix the left end: $x_0 = 0$
- Assume the initial discrete derivative $\Delta_h x_0 = \frac{x_1 - x_0}{h_0} = \frac{x_1}{h_0}$ (choose x_1)
- March forward, solving the recurrence for x_{k+2}:

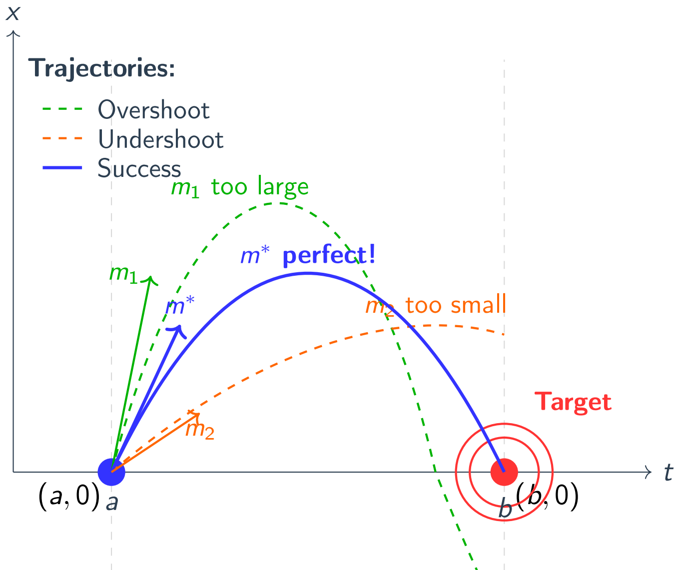
$$x_{k+2} = \left(1 + \frac{h_{k+1}\, r_k}{h_k\, r_{k+1}} - \frac{h_{k+1} h_k}{r_{k+1}} [\lambda w_k - q_k]\right) x_{k+1} - \frac{h_{k+1}\, r_k}{h_k\, r_{k+1}}\, x_k$$

- Tune x_1 until $x_{n+1} = 0$

The limitation

- x_k can overflow before reaching 0 $(10^5,\ 10^{28}, \dots) \Rightarrow$ Target $x_{n+1} = 0$ becomes unfindable

Visual of Regular Shooting Method



Algorithm:

1. Start at $(a, 0)$
2. Guess slope m
3. Solve IVP
4. Check if $x(b) = 0$
5. Adjust m if needed
6. Repeat until target hit

Prüfer Substitution: Phase and Amplitude

Test problem

$$\Delta_h\left(\Delta_h x_k\right) + \lambda\, x_{k+1} = 0, \qquad x_0 = x_{n+1} = 0$$

Prüfer substitution: Cartesian to Polar

$$x_k = \mu_k \sin\theta_k, \qquad \Delta_h x_k = \mu_k \cos\theta_k, \qquad \mu_k > 0 \qquad 0 \le \theta_k < 2\pi$$

Resulting equations

Phase:

$$\sin\left(h_k \Delta_h \theta_k\right) = h_k\left(\cos\theta_k \cos\theta_{k+1} + \lambda\, h_k \cos\theta_k \sin\theta_{k+1} + \lambda\, \sin\theta_k \sin\theta_{k+1}\right)$$

Amplitude:

$$\Delta_h \mu_k = -\mu_k\left(\frac{h_k}{2}\left[(\Delta_h \sin\theta_k)^2 + (\Delta_h \cos\theta_k)^2\right] - \sin\theta_{k+1}\cos\theta_k + \lambda\, h_k \cos\theta_{k+1}\cos\theta_k + \lambda\, \cos\theta_{k+1}\sin\theta_k\right)$$

$\langle * \rangle$ Phase equation is independent of amplitude.

Amplitude vs. Phase: Boundedness

Amplitude μ_k

$$\mu_k = \sqrt{x_k^2 + (\Delta_h x_k)^2} = \sqrt{x_k^2 + (x_{k+1} - x_k)^2}$$

No bound: μ_k can grow without limit as x_k grows.

Phase θ_k

$$\theta_k = \operatorname{arccot}\left(\frac{\Delta_h x_k}{x_k}\right)$$

Since $\mu_k > 0$, numerator ($\sin\theta_k$) and denominator ($\cos\theta_k$) never vanish together, so θ_k is always defined. As arccot,

$$\theta_k \in (0, \pi).$$

Bounded: θ_k cannot grow without limit as x_k grows.

Uniform Grid

Outline

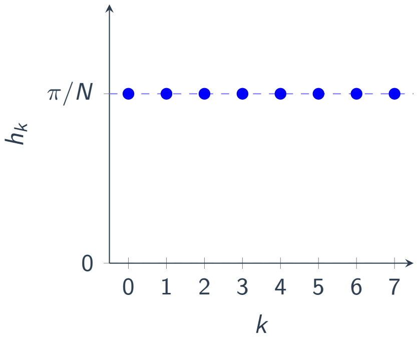
- Equal subintervals; the reference grid.
- One parameter: N, the number of subintervals.

Partition $[0, \pi]$ by

$$\text{Node } t_k = \frac{k\,\pi}{N}, \qquad \text{Spacing } h_k = \frac{\pi}{N},$$

for $k = 0, 1, \ldots, N$.

Step sizes h_k vs. k



Every step equal: a flat line at $h_k = \pi/N$.

Uniform Grid Eigenvalue Comparison

10 eigenvalues				Selected larger eigenvalues			
k	$\lambda_k = k^2$	λ_k^{reg}	λ_k^{Pr}	k	$\lambda_k = k^2$	λ_k^{reg}	λ_k^{Pr}
1	1	1.0000000162	1.0000000000	10	100	100.0156730119	100.0000000000
2	4	4.0000010376	4.0000000000	20	400	400.9018627489	400.0000000000
3	9	9.0000117977	9.0000000000	30	900	908.5147455464	900.0000000000
4	16	16.0000661240	16.0000000001	40	1600	1634.7863178329	1600.0000000000
5	25	25.0002514447	25.0000000000	50	2500	2569.4022595094	2500.0000000000
6	36	36.0007479063	36.0000000000	60	3600	3597.2162582778	3600.0000000000
7	49	49.0018773256	49.0000000001				
8	64	64.0041609969	64.0000000000				
9	81	81.0083851914	81.0000000000				
10	100	100.0156730119	100.0000000000				

Prüfer shooting consistently produces accurate approximations of eigenvalues, while regular shooting does so with some errors.

Visual of Uniform Grid

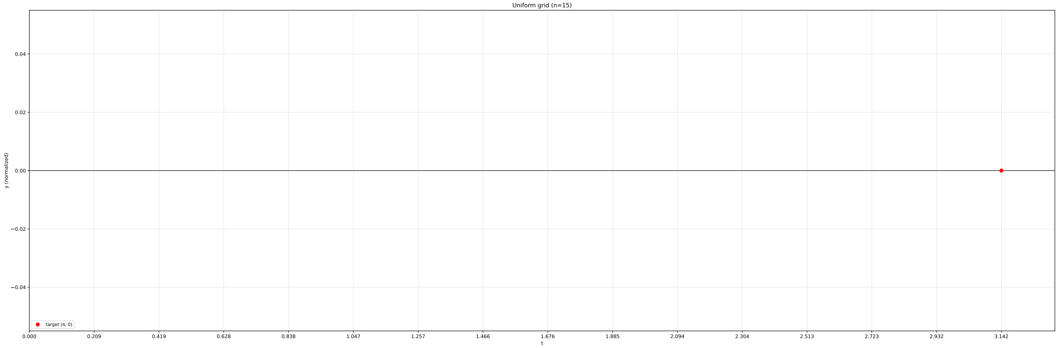


Figure 1: Uniform grid

Outline

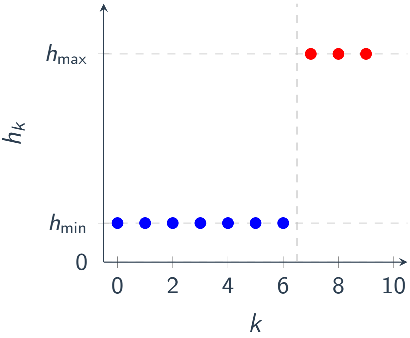
- Two uniform blocks joined at a break point.
- Parameters: σ_p (point fraction) and σ_d (domain fraction).
- Working values: $\sigma_p = 0.7$, $\sigma_d = 0.3$.
- Spacing ratio $h_{\max}/h_{\min} \approx 5.4$.

Definition

With $N_1 = \lfloor \sigma_p N \rceil$ *and* $N_2 = N - N_1$,

$$h_k = \begin{cases} \dfrac{\sigma_d \pi}{N_1}, & 0 \le k < N_1, \\ \dfrac{(1-\sigma_d)\pi}{N_2}, & N_1 \le k < N. \end{cases}$$

Step sizes h_k vs. k



$N = 10$: seven small steps, then three large. The jump is the ratio ≈ 5.4.

Clustered Grid Eigenvalue Comparison (70% points in first 30%)

10 eigenvalues				Selected larger eigenvalues			
k	$\lambda_k = k^2$	λ_k^{reg}	λ_k^{Pr}	k	$\lambda_k = k^2$	λ_k^{reg}	λ_k^{Pr}
1	1	1.0000003364	1.0000000000	10	100	100.2768060274	100.0000000000
2	4	4.0000214048	4.0000000000	20	400	407.7564289865	400.0000000000
3	9	9.0002414825	9.0000000000	30	900	840.3700687103	900.0000000000
4	16	16.0013386422	16.0000000001	40	1600	1116.1372616128	1600.0000000000
5	25	25.0050185147	25.0000000000	50	2500	1358.3033745351	2499.9999999999
6	36	36.0146687647	36.0000000000	60	3600	1999.5096864725	3600.0000000000
7	49	49.0360619007	49.0000000001				
8	64	64.0780124154	64.0000000000				
9	81	81.1528810054	81.0000000000				
10	100	100.2768060275	100.0000000000				

Prüfer shooting consistently produces accurate approximations of eigenvalues, while regular shooting fails to accurately compute larger eigenvalues.

Visual of Clustered Grid

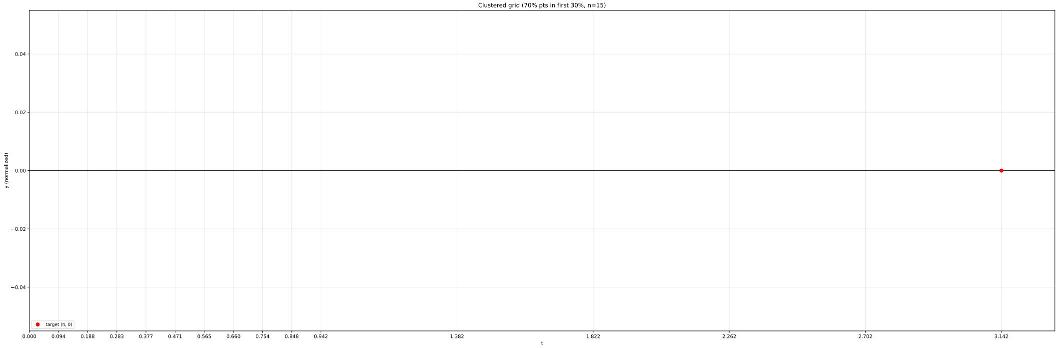


Figure 2: Clustered grid

Graded Grid

Outline

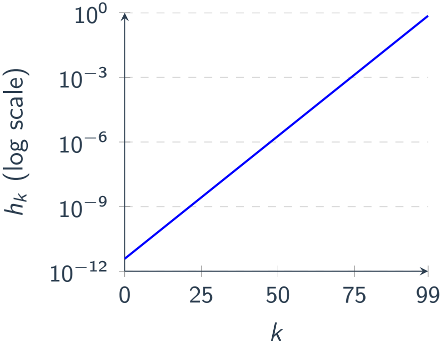
- Geometric progression of step sizes.
- One parameter: the ratio $r > 0$.
- Working value: $r = 1.3$ at $N = 100$.
- Spacing ratio $h_{\max}/h_{\min} = r^{N-1} \approx 1.9 \times 10^{11}$.

Definition

Let $h_k = h_0\, r^k$ with h_0 chosen so that $\sum_{k=0}^{N-1} h_k = \pi$:

$$h_0 = \frac{(r-1)\,\pi}{r^N - 1}, \qquad t_{k+1} = t_k + h_k.$$

Step sizes h_k vs. k



Geometric growth $\Rightarrow$ straight line on a log axis.

The 11-decade rise is the ratio $\approx 1.9 \times 10^{11}$.

Graded Grid Eigenvalue Comparison (ratio $= 1.3$)

10 eigenvalues				Selected larger eigenvalues			
k	$\lambda_k = k^2$	λ_k^{reg}	λ_k^{Pr}	k	$\lambda_k = k^2$	λ_k^{reg}	λ_k^{Pr}
1	1	1.0012290593	1.0000000000	10	100	84.0272562384	100.0000000000
2	4	4.0416450577	4.0000000000	15	225	335.6984898946	225.0000000000
3	9	8.8312738207	9.0000000000	20	400	1309.7276468120	400.0000000000
4	16	13.0750598684	16.0000000001	21	441	1702.3954734796	441.0000000000
5	25	18.5934352481	25.0000000000	22	484	2234.0226888491	484.0000000000
6	36	25.2962038872	36.0000000000	23	529	2904.0757565460	529.0000000000
7	49	34.5418828258	49.0000000001	30	900	—	900.0000000000
8	64	46.7784765146	64.0000000000	40	1600	—	1600.0000000000
9	81	62.2179518992	81.0000000000	50	2500	—	2499.9999999999
10	100	84.0272562384	100.0000000000	60	3600	—	3600.0000000000

Prüfer shooting remains consistent on graded grids, while regular shooting fails to finish its computation of larger eigenvalues.

Visual of Graded Grid

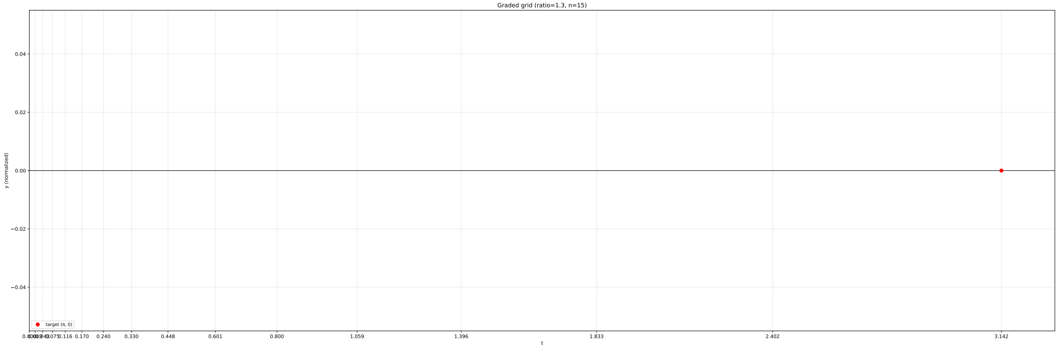


Figure 3: Graded grid

What we learned

The big take aways

- Amplitude has no bound and can overshoot and undershoot easily
- Prüfer substitution conditioned Phase Equation to be free of Amplitude
- Therefore allow Prüfer based shooting to be more robust than Regular shooting on both uniform and non-uniform grids
- The n^{th} eigen function has $n - 1$ eigen zeros

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Prüfer based shooting algorithm (uniform discrete)

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Questions?

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