EIGENVALUE PROBLEMS

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1. Eigenvalues of a linear operator

A linear operator on a space of functions is a map L satisfying $L(\alpha u + \beta v) = \alpha Lu + \beta Lv$, of which differentiation is the basic example. An eigenvalue records an input on which the operator acts by scaling.

Definition 1.1. Let V be a vector space of functions and $L : V \to V$ a linear operator. A scalar λ is an *eigenvalue* of L with *eigenfunction* $u \neq 0$ if

$$Lu = \lambda u.$$

A differential operator considered on all functions typically has a continuum of eigenvalues. For $L = \dfrac{d^2}{dx^2}$, the equation $u'' = \lambda u$ has the nonzero solution $u = e^{\sqrt{\lambda}x}$ for every λ, so every real number is an eigenvalue. A discrete set of eigenvalues is obtained by admitting only those functions that satisfy prescribed homogeneous conditions at the endpoints of an interval.

Definition 1.2. An *eigenvalue problem* for L consists of the equation $Lu = \lambda u$ on an interval together with homogeneous boundary conditions. A scalar λ for which a nontrivial u satisfies both is an *eigenvalue* of the problem, and u is an associated *eigenfunction*.

2. The model problem

Consider the eigenvalue problem

$$-u'' = \lambda u \quad \text{on } (0, L), \qquad u(0) = 0 = u(L). \tag{1}$$

Its eigenvalues and eigenfunctions are determined in the following theorem; the argument splits according to the sign of λ, which governs the character of the equation.

Theorem 2.1. *The eigenvalues of* (1) *are*

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, \ldots,$$

each with the one-dimensional space of eigenfunctions spanned by

$$u_n(x) = \sin\frac{n\pi x}{L}.$$

Proof. Suppose $\lambda < 0$ and write $\lambda = -\mu^2$ with $\mu > 0$. Then $u'' = \mu^2 u$, with general solution $u = A\cosh\mu x + B\sinh\mu x$. The condition $u(0) = 0$ gives $A = 0$, and $u(L) = B\sinh\mu L = 0$ then gives $B = 0$, since $\sinh\mu L \neq 0$. Only the trivial solution exists, so there is no negative eigenvalue.

Suppose $\lambda = 0$. Then $u'' = 0$, so $u = ax + b$; the conditions $u(0) = 0$ and $u(L) = 0$ give $b = 0$ and $a = 0$. Thus 0 is not an eigenvalue.

Suppose $\lambda > 0$ and write $\lambda = \omega^2$ with $\omega > 0$. The general solution of $u'' = -\omega^2 u$ is $u = A\cos\omega x + B\sin\omega x$. The condition $u(0) = 0$ gives $A = 0$, leaving $u = B\sin\omega x$. The condition $u(L) = B\sin\omega L = 0$ admits a nontrivial solution if and only if $\sin\omega L = 0$, that is, $\omega L = n\pi$ for a positive integer n. Hence $\omega = n\pi/L$, the eigenvalues are $\lambda_n = \omega^2 = (n\pi/L)^2$, and the corresponding eigenfunctions are $u_n = \sin(n\pi x/L)$. □

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Remark. The eigenvalues λ_n are real and positive and increase without bound, and the eigenfunctions oscillate increasingly with n: u_n has exactly $n-1$ zeros in $(0, L)$ (Figure 1). For $L = \pi$ one has $\lambda_1 = 1$ and $u_1 = \sin x$, consistent with the fact that π is the least length on which $u'' + u = 0$ admits a nontrivial solution vanishing at both endpoints.

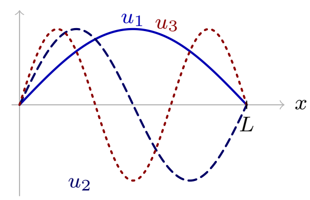

FIGURE 1. The first three eigenfunctions $u_n = \sin(n\pi x/L)$ on $[0, L]$. The n-th has $n-1$ interior zeros: u_1 none, u_2 one, u_3 two.

3. Sturm–Liouville eigenvalue problems

The three properties exhibited by the model problem—real eigenvalues, orthogonality of eigenfunctions belonging to distinct eigenvalues, and increasing oscillation—are not particular to the sine. They hold for the class of Sturm–Liouville problems, which is the general form of a second-order self-adjoint eigenvalue problem. Orthogonality of the eigenfunctions underlies the expansion of a function in a series of eigenfunctions, of which the Fourier sine series is the instance arising from the model problem.

Definition 3.1. A *regular Sturm–Liouville eigenvalue problem* on $[a, b]$ is

$$-\big(p(x)u'\big)' + q(x)u = \lambda\, w(x)\, u \quad \text{on } (a, b), \tag{2}$$

where $p \in C^1$ with $p > 0$, and q, w are continuous with $w > 0$, together with *separated* boundary conditions

$$\alpha_1 u(a) + \alpha_2 u'(a) = 0, \qquad \beta_1 u(b) + \beta_2 u'(b) = 0,$$

with $(\alpha_1, \alpha_2) \neq (0, 0)$ and $(\beta_1, \beta_2) \neq (0, 0)$. The positive function w is the *weight.*

The model problem (1) is the case $p = 1$, $q = 0$, $w = 1$ on $[0, L]$ with $\alpha_2 = \beta_2 = 0$. Set

$$Lu = -\big(pu'\big)' + qu,$$

so that (2) is the equation $Lu = \lambda wu$. The operator L is the general formally self-adjoint second-order differential operator. Throughout, $\langle u, v\rangle = \int_a^b u\, v\, \mathrm{d}x$ denotes the standard inner product.

Lemma 3.2 (Lagrange identity)**.** *For twice-differentiable u, v,*

$$v\, Lu - u\, Lv = -\big[p\,(u'v - uv')\big]'.$$

If, in addition, u and v satisfy the separated boundary conditions, then $\langle Lu, v\rangle = \langle u, Lv\rangle$.

Proof. The q-terms cancel, so $v\, Lu - u\, Lv = -v(pu')' + u(pv')'$. Differentiation of the product on the right gives $u(pv')' - v(pu')' = \big[p(uv' - u'v)\big]'$, whence

$$v\, Lu - u\, Lv = \big[p(uv' - u'v)\big]' = -\big[p(u'v - uv')\big]'.$$

Integrating over $[a, b]$,

$$\langle Lu, v\rangle - \langle u, Lv\rangle = \int_a^b (v\, Lu - u\, Lv)\, \mathrm{d}x = \big[p(uv' - u'v)\big]_a^b.$$

At $x = a$ the nonzero pair (α_1, α_2) satisfies the homogeneous system $\alpha_1 u(a) + \alpha_2 u'(a) = 0$, $\alpha_1 v(a) + \alpha_2 v'(a) = 0$; a homogeneous 2×2 system with a nontrivial solution has vanishing determinant, so

$u(a)v'(a) - u'(a)v(a) = 0$. The same holds at b. Hence the boundary term vanishes and $\langle Lu, v\rangle = \langle u, Lv\rangle$. $\square$

The identity $\langle Lu, v\rangle = \langle u, Lv\rangle$, valid for u, v satisfying the boundary conditions, states that L is *self-adjoint*. Reality of the eigenvalues and orthogonality of the eigenfunctions are consequences of self-adjointness, established in the next two theorems; the weight w enters the orthogonality relation.

Theorem 3.3 (Reality)**.** *Every eigenvalue of* (2) *is real.*

Proof. Let λ be an eigenvalue with eigenfunction $u \neq 0$, regarded as possibly complex. Taking the inner product of $Lu = \lambda w u$ with $\bar{u}$ and applying Lemma 3.2, which holds since L has real coefficients,

$$\lambda \int_a^b w\,|u|^2\,\mathrm{d}x = \langle Lu, \bar{u}\rangle = \langle u, L\bar{u}\rangle = \bar{\lambda} \int_a^b w\,|u|^2\,\mathrm{d}x.$$

Since $w > 0$ and $u \neq 0$, the integral $\int_a^b w\,|u|^2$ is positive. Therefore $\lambda = \bar{\lambda}$, and λ is real. $\square$

Define the *weighted inner product*

$$\langle u, v\rangle_w = \int_a^b u(x)\,v(x)\,w(x)\,\mathrm{d}x.$$

Theorem 3.4 (Orthogonality)**.** *If u_n, u_m are eigenfunctions of* (2) *for distinct eigenvalues $\lambda_n \neq \lambda_m$, then $\langle u_n, u_m\rangle_w = 0$.*

Proof. By Lemma 3.2, $\langle Lu_n, u_m\rangle = \langle u_n, Lu_m\rangle$. Substituting $Lu_n = \lambda_n w u_n$ and $Lu_m = \lambda_m w u_m$ yields

$$(\lambda_n - \lambda_m) \int_a^b u_n\,u_m\,w\,\mathrm{d}x = 0.$$

Since $\lambda_n \neq \lambda_m$, it follows that $\int_a^b u_n u_m w\,\mathrm{d}x = 0$. $\square$

For the model problem $w = 1$, and the relation reduces to $\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\,\mathrm{d}x = 0$ for $n \neq m$. The following theorem, stated without proof, describes the full spectrum.

Theorem 3.5 (Sturm)**.** *A regular Sturm–Liouville problem has infinitely many eigenvalues, all real and simple, forming an increasing sequence*

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots \longrightarrow \infty.$$

The eigenfunction u_n belonging to λ_n has exactly $n-1$ zeros in the open interval (a, b).

Remark. Theorem 3.5 has two parts. The eigenvalues form a discrete increasing sequence tending to infinity, so there is a least eigenvalue and no finite accumulation point. The eigenfunctions gain one interior zero at each step: u_n has exactly one more zero in (a, b) than u_{n-1}. For the model problem this is realized by $u_n = \sin(n\pi x/L)$ (Figure 1). The monotone increase in the number of zeros is the subject of oscillation theory.

4. Exercises

Exercise 4.1. Solve the Neumann problem $-u'' = \lambda u$ on $(0, \pi)$ with $u'(0) = 0 = u'(\pi)$, and show that the eigenvalues are $\lambda_n = n^2$ for $n = 0, 1, 2, \ldots$ with eigenfunctions $\cos(nx)$. Note that $\lambda = 0$ is an eigenvalue, with constant eigenfunction, in contrast to the Dirichlet problem (1).

Exercise 4.2. Verify that $\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\,\mathrm{d}x = 0$ for positive integers $n \neq m$, and compute the value of the integral when $n = m$.

Exercise 4.3. For the model problem (1), determine the interior zeros of u_n explicitly and confirm that there are $n-1$ of them, in agreement with Theorem 3.5.

Exercise 4.4. Put each equation into Sturm–Liouville form $-(pu')' + qu = \lambda wu$, identifying p, q, and w.

(a) $-u'' = \lambda u$.
(b) $u'' + u' + \lambda u = 0$. *(Multiply by e^x and recognize $(e^x u')'$.)*

Exercise 4.5. Show that the operator $Lu = -u''$ with Dirichlet conditions has only positive eigenvalues: multiply $-u'' = \lambda u$ by u, integrate over $[0, L]$, integrate by parts using $u(0) = u(L) = 0$, and conclude that $\lambda = \dfrac{\int_0^L (u')^2}{\int_0^L u^2} > 0$ for nontrivial u.