FUNCTION SPACES

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Guided Problems for Homework

Work through these problems together. They are short and meant to build the techniques you will use in the homework.

Problem 1. Check whether $V = \{f \in C[0,1] : f(0) = 0\}$ is a function space. Verify each of the three conditions of the Subspace Test.

Problem 2. Show that the set $\{f \in C[0,1] : f(x) \geq 0 \text{ for all } x \in [0,1]\}$ is not a function space by exhibiting one specific counterexample.

Problem 3. Is $\{f \in C^1[0,1] : f(0) = 0 \text{ and } f'(0) = 0\}$ a function space? Justify.

Problem 4. Let $V_1 = \{f \in C[0,1] : f(0) = 0\}$ and $V_2 = \{f \in C[0,1] : f(1) = 0\}$. Show that $V_1 \cap V_2$ contains the function $f(x) = x(1-x)$, and explain in one sentence why $V_1 \cap V_2$ is a function space.

Problem 5. Let $\phi_1(x) = \sin x$ and $\phi_2(x) = \cos x$. Write down three specific functions in $\operatorname{span}\{\phi_1, \phi_2\}$. Is $\sin(2x)$ in this span?

Problem 6. Suppose $a_0 + a_1 x + a_2 x^2 = 0$ for all $x \in [0,1]$. By evaluating at $x = 0$, differentiating and evaluating at $x = 0$, and differentiating once more and evaluating at $x = 0$, conclude that $a_0 = a_1 = a_2 = 0$.

Problem 7. Find a specific nonzero function $f \in C^1[0,1]$ with $f(0) = f(1) = 0$.

Problem 8. Show that the operator $T : C^1[0,1] \to \mathbb{R}$ defined by $T(f) = f'(0) + f(1)$ is linear.

Problem 9. Verify that $\{1, x\}$ is a basis for $\mathcal{P}_1[0,1]$. What is $\dim \mathcal{P}_1[0,1]$?

Problem 10. Let $f \in C[0,1]$ with $|f(x)| \leq 1$ for all $x \in [0,1]$. Suppose $\int_0^1 f(x)(x^2 - x)\,dx = 0$. Using this hypothesis, deduce a relationship between $\int_0^1 f(x)\,x^2\,dx$ and $\int_0^1 f(x)\,x\,dx$.

Problem 11. Compute $\|f\|_1$, $\|f\|_2$, and $\|f\|_\infty$ for $f(x) = x$ on $[0,1]$. Verify that $\|f\|_1 \leq \|f\|_2 \leq \|f\|_\infty$.

Problem 12. Let $f \in C[0,1]$ with $f(x) \geq 0$ for all x. Suppose $f(1/2) > 0$. Explain in one or two sentences why $\int_0^1 f(x)\,dx > 0$.

Problem 13. Apply the Cauchy–Schwarz inequality to the functions $f(x) = x$ and $g(x) = 1$ on $[0,1]$. What inequality do you obtain?

Problem 14. Compute $\int_0^\pi \sin(x)\cos(x)\,dx$. Are $\sin(x)$ and $\cos(x)$ orthogonal in $L^2[0,\pi]$?

Problem 15. Is the sequence $a_n = 1/n$ in ℓ^2? Is it in ℓ^1? Justify each answer.

Homework Exercises

Write your solutions in LaTeX. All proofs should be complete and rigorous. Justify every step, including standard facts from calculus and linear algebra that you invoke. You may use any result proved in the lecture notes.

Problem 1. Determine whether each of the following sets is a function space on $[0,1]$. Prove your answer in each case.

(a) $\{f \in C[0,1] : f(0) = f(1)\}$.
(b) $\{f \in C[0,1] : \int_0^1 f(x)\,dx = 0\}$.

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(c) $\{f \in C[0,1] : f(x) \geq f(0) \text{ for all } x \in [0,1]\}$.
(d) $\{f \in C^1[0,1] : f(0) = 0 \text{ and } f'(0) = 1\}$.

Problem 2. Let V_1 and V_2 be function spaces on $[a,b]$.

(a) Prove that $V_1 \cap V_2$ is a function space.
(b) Find specific function spaces $V_1, V_2 \subseteq C[0,1]$ for which $V_1 \cup V_2$ is *not* a function space, and justify your example.
(c) Define $V_1 + V_2 := \{f_1 + f_2 : f_1 \in V_1, f_2 \in V_2\}$. Prove that $V_1 + V_2$ is a function space.

Problem 3. Let $\phi_1, \phi_2, \ldots, \phi_k \in \mathbb{R}[a,b]$. Define the *span* of $\phi_1, \ldots, \phi_k$ by

$$\operatorname{span}\{\phi_1, \ldots, \phi_k\} := \left\{ \sum_{j=1}^{k} \alpha_j \phi_j : \alpha_1, \ldots, \alpha_k \in \mathbb{R} \right\}.$$

Prove that $\operatorname{span}\{\phi_1, \ldots, \phi_k\}$ is a function space on $[a,b]$.

Problem 4. Recall that $\mathcal{P}_n[a,b]$ denotes the function space of polynomials of degree at most n on $[a,b]$.

(a) Prove that the functions $1, x, x^2, \ldots, x^n$ are linearly independent in $\mathbb{R}[a,b]$ for any nondegenerate interval $[a,b]$.
(b) Conclude that $\dim \mathcal{P}_n[a,b] = n + 1$.

Problem 5. Prove that $C[a,b]$ is infinite-dimensional. That is, show that for every integer $n \geq 0$, there exist $n+1$ linearly independent functions in $C[a,b]$. You may use the result of Problem 4.

Problem 6. Let V be a function space on $[a,b]$ and let $c \in [a,b]$ be fixed. Define

$$V_0 := \{f \in V : f(c) = 0\}.$$

(a) Prove that V_0 is a function space.
(b) Prove that if V contains a function g with $g(c) \neq 0$, then $V_0 \neq V$; that is, V_0 is a proper subspace of V.
(c) For $V = C[0,1]$ and $c = 1/2$, exhibit two specific functions $f_1, f_2 \in V$ with $f_1 \in V_0$ and $f_2 \notin V_0$.

Problem 7. Define

$$V := \{f \in C^1[a,b] : f(a) = f(b) = 0\}.$$

(a) Prove that V is a function space on $[a,b]$.
(b) Prove that the differentiation operator $D : V \to C[a,b]$ defined by $D(f) = f'$ is linear.
(c) In the case $[a,b] = [0,1]$, find a nonzero element $f \in V$ and verify that it satisfies the boundary conditions.

Problem 8. Let $W \subseteq \mathbb{R}[a,b]$ be a finite-dimensional function space.

(a) Prove that W has a basis of size $\dim(W)$.
(b) Prove that if $\phi_1, \ldots, \phi_k$ are linearly independent elements of W with $k = \dim(W)$, then $\{\phi_1, \ldots, \phi_k\}$ is a basis of W.

Problem 9. Let V be a function space on $[a,b]$ and let $T : V \to V$ be a linear operator. Define

$$\ker(T) := \{f \in V : T(f) = 0\}, \qquad \operatorname{im}(T) := \{T(f) : f \in V\}.$$

Prove that both $\ker(T)$ and $\operatorname{im}(T)$ are function spaces on $[a,b]$.

Problem 10. Let $f \in C[0,1]$ with $f(x) \geq 0$ for all $x \in [0,1]$. Suppose

$$\int_0^1 f(x)\,dx = 0.$$

Prove that $f \equiv 0$ on $[0,1]$.

Problem 11. Let $f \in C[a,b]$. Prove that if

$$\int_a^b f(x)g(x)\,dx = 0 \qquad \text{for every } g \in C[a,b],$$

then $f \equiv 0$ on $[a,b]$.

Problem 12. Let $f \in C[0,1]$ and suppose $\int_0^1 f(x)\,dx = 0$. Prove that there exists $c \in (0,1)$ such that $f(c) = 0$.

Problem 13. Prove that for every $f \in C[0,1]$,

$$\left(\int_0^1 f(x)\,dx\right)^2 \leq \int_0^1 f(x)^2\,dx.$$

Identify the conditions under which equality holds.

Problem 14. Let $f \in C[a,b]$. Prove that

$$\|f\|_1 \leq \sqrt{b-a} \cdot \|f\|_2 \leq (b-a) \cdot \|f\|_\infty.$$

Conclude that $C[a,b] \subseteq L^2[a,b] \subseteq L^1[a,b]$.

Problem 15. Let $f, g \in C[a,b]$ with $f, g \geq 0$. Prove that

$$\left(\int_a^b \sqrt{f(x)g(x)}\,dx\right)^2 \leq \int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx.$$

Problem 16. Let V be a function space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|f\| = \sqrt{\langle f, f \rangle}$. Prove the parallelogram law:

$$\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2.$$

Problem 17. Let $T : V \to V$ be a linear operator on a function space with inner product, and assume T is *self-adjoint*: $\langle Tf, g \rangle = \langle f, Tg \rangle$ for all $f, g \in V$. Suppose

$$Ty_1 = \lambda_1 y_1, \qquad Ty_2 = \lambda_2 y_2,$$

with $\lambda_1 \neq \lambda_2$ and $y_1, y_2 \neq 0$. Prove that $\langle y_1, y_2 \rangle = 0$.

Problem 18. Compute $\int_0^\pi \sin(mx)\cos(nx)\,dx$ for positive integers m, n. Conclude that $\sin(mx)$ and $\cos(nx)$ are orthogonal in $L^2[0,\pi]$ for all such m, n.

Problem 19. For each $\alpha > 0$, determine for which $p \in [1, \infty]$ the sequence $a_n = 1/n^\alpha$ belongs to ℓ^p. Prove your answer.

Problem 20. Let $(a_n) \in \ell^2$. Prove that the series $\sum_{n=1}^\infty a_n/n$ converges absolutely.

Hint: apply Cauchy–Schwarz to the sequences (a_n) *and* $(1/n)$*.*