Bifurcation in Competing and Cooperating Species Models with Nonlinear Boundary Coupling

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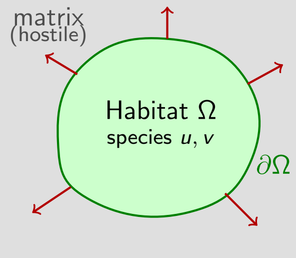
Ecological Motivation: Why Boundary Interactions?



Individuals can emigrate at $\partial\Omega$

Habitats are fragmented. Species live in patches Ω surrounded by a hostile matrix (roads, agriculture, urban land, . . .).

The boundary is a dispersal interface. Classical Robin BC:

$$\frac{\partial w}{\partial \eta} + \sqrt{\lambda}\,\gamma\, w = 0 \quad \text{on } \partial\Omega,$$

with $\gamma =$ *constant* matrix hostility.

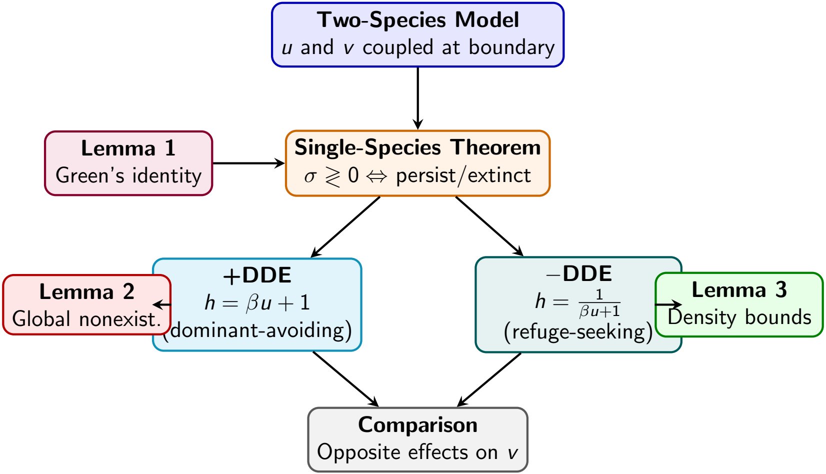
But emigration is trait-mediated. Animals sense conditions *at the edge* and adjust:

- Dominant-avoidance: neighbor ⇒ leave faster
- Refuge-seeking: neighbor ⇒ stay in patch

⇒ Boundary coefficient must depend on the neighbor:

$$\gamma \longrightarrow \gamma\, h(u, \beta).$$

Talk Overview



Two-Species Model
u and v coupled at boundary

Lemma 1
Green's identity

Single-Species Theorem
$\sigma \gtrless 0 \Leftrightarrow$ persist/extinct

Lemma 2
Global nonexist.

+DDE
$h = \beta u + 1$
(dominant-avoiding)

−DDE
$h = \frac{1}{\beta u + 1}$
(refuge-seeking)

Lemma 3
Density bounds

Comparison
Opposite effects on v

Main question: How does trait-mediated dispersal at the boundary affect coexistence?

The Model: Trait-Mediated Dispersal at Boundaries

Classical setting:

Reaction-diffusion on bounded domain $\Omega \subset \mathbb{R}^n$ with Robin BC:

$$-\Delta w = \lambda f(w) \quad \text{in } \Omega, \qquad \frac{\partial w}{\partial \eta} + \gamma w = 0 \quad \text{on } \partial\Omega$$

Well-known: there exists a critical λ^* such that

- $\lambda \leq \lambda^*$: only trivial solution
- $\lambda > \lambda^*$: unique positive steady state

Our extension: Boundary coefficient depends on another species' solution.

Species u (independent):

$$\boxed{\begin{aligned} -\Delta u &= \lambda u(1-u) \quad \text{in } \Omega \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda}\gamma_1 u &= 0 \quad \text{on } \partial\Omega \end{aligned}}$$

Species v (coupled to u):

$$\boxed{\begin{aligned} -\Delta v &= \lambda r v(1-v-bu) \quad \text{in } \Omega \\ \frac{\partial v}{\partial \eta} + \sqrt{\lambda}\gamma_2 h(u,\beta) v &= 0 \quad \text{on } \partial\Omega \end{aligned}}$$

(+DDE) Dominant-avoidance: $h(u,\beta) = \beta u + 1$ (increasing in u)

(-DDE) Refuge-seeking: $h(u,\beta) = \dfrac{1}{\beta u + 1}$ (decreasing in u)

Prior Work: Where This Sits

- **Nonlinear boundary conditions on bounded domains.**
  Amann (1976); Cantrell & Cosner, *JDE* 2006, *Bull. Math. Biol.* 2007; Cantrell–Cosner book (2003).
- **Patch–matrix derivation of $h(u, \beta)$.**
  Cronin, Goddard II, Shivaji, *Bull. Math. Biol.* 2019 — $h(u, \beta)$ derived from biased random walks at $\partial\Omega$; Maciel & Lutscher (2013); Ovaskainen (2003, 2004).
- **Single-species DDE on patches.**
  Harman, Goddard II, Shivaji, Cronin, *Amer. Nat.* 2020; Cronin, Fonseka, Goddard II, Leonard, Shivaji, *MBE* 2019; Fonseka et al., *DCDS* 2020 (U-shaped DDE).
- **Trait-mediated dispersal, multi-species.**
  Cronin, Goddard II, Muthunayake, Shivaji, *MBE* 2020 (mutualists); Cronin, Goddard II, Muthunayake, Quiroa, Shivaji, *J. Math. Biol.* 2024 (predator–prey, hump-shaped DAR).
- **Immediate predecessor.**
  Acharya, **Bandyopadhyay**, Cronin, Goddard II, Muthunayake, Shivaji, *Nonlinear Anal. RWA* 2023 — same two-species LV competition, constant emigration ($\beta = 0$).

This paper: $\beta > 0$, trait-mediated DDE at $\partial\Omega$ — full coexistence analysis for both $+$DDE and $-$DDE.

Important Terminology

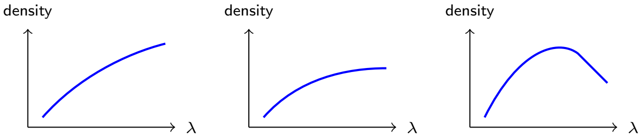
DDE: Density-Dependent Emigration

How does emigration rate change when dominant species u is present?

Type	Boundary function	Biological meaning
+DDE	$h(u, \beta) = \beta u + 1$	Avoid dominant ⇒ leave patch
−DDE	$h(u, \beta) = \dfrac{1}{\beta u + 1}$	Seek refuge ⇒ stay in patch

DAR: Density-Area Relationship

How does average population density $\int_\Omega v\, dx$ change as patch size (λ) increases?



Positive Neutral Hump-shaped

Why Study General Single-Species Dynamics?

Our two-species model:

$$-\Delta u = \lambda u(1-u) \qquad \frac{\partial u}{\partial \eta} + \sqrt{\lambda}\gamma_1 u = 0$$

$$-\Delta v = \lambda r v(1 - v - b\,\underbrace{u(x)}_{\text{known}}) \qquad \frac{\partial v}{\partial \eta} + \sqrt{\lambda}\gamma_2\,\underbrace{h(u(x),\beta)}_{\text{known}}\,v = 0$$

Once u **is solved:** v-equation has **spatially varying** coefficients!

$$-\Delta v = \lambda\,\underbrace{rv(1-v-bu(x))}_{f(\lambda,v,x)}, \quad \frac{\partial v}{\partial \eta} + \underbrace{\sqrt{\lambda}\gamma_2 h(u(x),\beta)}_{\mu(\lambda)(x)}\,v = 0$$

Need a general framework

Single-species dynamics with spatially varying $f(\lambda, w, x)$ and $\mu(\lambda)(x)$

General Framework: Single-Species Dynamics

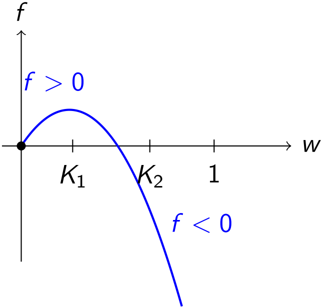
Single-species problem: $-\Delta w = \lambda f(\lambda, w, x)$ in Ω, $\frac{\partial w}{\partial \eta} + \mu(\lambda)(x)w = 0$ on $\partial\Omega$

Hypothesis (H1): f is smooth with logistic-type structure:

(a) $f(\lambda, 0, x) = 0$

(b) $\exists\, 0 < K_{1,\lambda} < K_{2,\lambda} \leq 1$:
$f > 0$ on $(0, K_{1,\lambda})$ $f < 0$ on $[K_{2,\lambda}, \infty)$

(c) $-m^* \leq f_{ww} \leq -M^* < 0$ (bounded concavity)



✓ Linearize
✓ $m(\lambda)(x) = f_w(\lambda, 0, x)$
✓ $\sigma(\lambda, m(\lambda), \mu(\lambda))$ Principal EV of:

$$-\Delta\phi - \lambda m(\lambda)(x)\phi = \sigma\phi \text{ in } \Omega, \quad \frac{\partial \phi}{\partial \eta} + \mu(\lambda)(x)\phi = 0 \text{ on } \partial\Omega$$

The sign of σ determines everything

- $\sigma \geq 0 \Rightarrow$ zero is stable (population dies out)
- $\sigma < 0 \Rightarrow$ zero is unstable (population can grow)

Verifying H1 for Our Model

For v-equation: $f(\lambda, v, x) = rv(1 - v - bu(x))$

(a) $f(\lambda, 0, x) = 0$ ✓

(b) $f > 0$ for small v, $f < 0$ for large v?

- $f = rv(1 - bu(x) - v)$
- $f > 0$ for small v requires $1 - bu(x) > 0$
- Since $u < 1$ on Ω: $b < 1 \Rightarrow$ always satisfied ✓
- $b > 1$ and u large $\Rightarrow 1 - bu(x) < 0 \Rightarrow$ H1(b) **fails!**

(c) $f_{vv} = -2r < 0$ ✓ $(m^* = M^* = 2r)$

Consequences

$b < 1$:	H1 (b) holds $\Rightarrow$ Single Species Theorem (coming up!) gives existence
$b > 1$, small λ:	$u \approx 0 \Rightarrow$ H1 (b) holds $\Rightarrow$ existence for bounded λ-range
$b > 1$, large λ:	$u \to 1 \Rightarrow$ H1 (b) fails $\Rightarrow$ nonexistence (Theorem uses H1 (a),(c))

Stability via Principal Eigenvalue

Consider stability in the **Lyapunov sense** (Pao, 1992):

Stable	Start close ⇒ stay close
Asymptotically stable	Start close ⇒ stay close ⇒ converge
Globally asymptotically stable	Converge from any initial condition
Unstable	Nearby solutions can escape

EVP for Linear BC:

$$\boxed{\begin{aligned} -\Delta\phi - \lambda m\phi &= \sigma\phi \text{ in } \Omega \\ \frac{\partial\phi}{\partial\eta} + \mu\phi &= 0 \text{ on } \partial\Omega \end{aligned}}$$

EVP for Nonlinear BC:

$$\boxed{\begin{aligned} -\Delta\phi - \lambda m\phi &= \tilde{\sigma}\phi \text{ in } \Omega \\ \frac{\partial\phi}{\partial\eta} + \mu\phi &= \tilde{\sigma}\phi \text{ on } \partial\Omega \end{aligned}}$$

Lemma (Local Stability —Goddard & Shivaji, 2017)

$\tilde{\sigma} \geq 0 \Rightarrow$ *trivial solution is stable;* $\tilde{\sigma} < 0 \Rightarrow$ *trivial solution is unstable*

Observe (Goddard, Morris, Robinson & Shivaji, 2018)

$\operatorname{sgn}(\sigma) = \operatorname{sgn}(\tilde{\sigma})$, so we work with the simpler problem on the left.

Theorem: Complete Dynamics of Single Species

Theorem

Suppose (H1) holds. Then:

(A) *If* $\sigma(\lambda, m(\lambda), \mu(\lambda)) \geq 0$*:*

- $w \equiv 0$ *is globally asymptotically stable* $\Rightarrow$ *No positive solution exists*

(B) *If* $\sigma(\lambda, m(\lambda), \mu(\lambda)) < 0$*:*

- $w \equiv 0$ *is unstable* $\Rightarrow$ *There exists a* **unique** *globally asymptotically stable positive solution* $w(\lambda)$
- $w(\lambda)(x) \in (0,1)$ *for all* $x \in \Omega$
- $w(\lambda) \to 0^+$ *as* $\sigma \to 0^-$ *(bifurcation from trivial solution)*
- $\mu_1 \leq \mu_2 \Rightarrow w(\mu_1) \geq w(\mu_2)$ *(decreasing in boundary loss)*

	Lemma	**Theorem**
Scope	Local (near zero)	Global (all initial data)
$\sigma \geq 0$	Zero is stable	Zero is GAS, no positive solution
$\sigma < 0$	Zero is unstable	Unique positive solution, GAS
Extra info	None	Bounds, bifurcation, monotonicity

Proof of Theorem: Part A (Nonexistence)

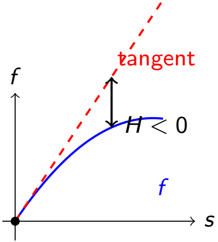
Goal: $\sigma \geq 0 \Rightarrow$ no positive solution exists.
Setup: Assume $w > 0$ exists. Pair with eigenfunction $\phi > 0$ (same BC).
Green's Second Identity:

$$\int_\Omega [(-\Delta w)\phi + (\Delta\phi)w]\,dx = \int_{\partial\Omega}\left[-\frac{\partial w}{\partial\eta}\phi + \frac{\partial\phi}{\partial\eta}w\right]ds$$

LHS (Substitute PDEs) $=$

$$\int_\Omega \phi\left[\lambda\underbrace{(f(\lambda,w,x) - f_w(\lambda,0,x)w)}_{=:H(w,x)} - \sigma w\right]dx$$

Why $H < 0$?



Concavity: tangent above graph

Contradiction

$\phi, \lambda, w > 0$, $\quad H(w,x) < 0$, $\quad \sigma \geq 0$
$\Rightarrow$ LHS < 0 but RHS $= 0$ (Boundary terms cancel)

Global stability: ($\sigma \geq 0$ + Local Stability Lemma $\Rightarrow$ trivial solution is stable locally)
$Z \equiv N$ for $N > 1$ is supersolution: by (H1), $K_{2,\lambda} \leq 1 \Rightarrow f < 0$ for $w > 1$
Local stability of 0 + No other positive solution + global supersolution $\Rightarrow$ Globaly Asymptotically Stable trivial solution (Pao, 1992)

Part B: Construction of Sub/Supersolutions

Goal: Construct sub and supersolutions using $\psi = \delta\phi$ where $\phi > 0$ is the eigenfunction.

Boundary: $\dfrac{\partial \psi}{\partial \eta} + \mu\psi = \delta\left(\dfrac{\partial \phi}{\partial \eta} + \mu\phi\right) = 0 \quad$ (since ϕ satisfies same BC)

Interior: Compute $I := -\Delta\psi - \lambda f(\lambda, \psi, x)$

$$\begin{aligned} I &= \delta\sigma\phi + \delta\lambda f_w(\lambda, 0, x)\phi - \lambda f(\lambda, \delta\phi, x) \\ &= \delta\sigma\phi - \lambda\left[f(\lambda, \delta\phi, x) - f_w(\lambda, 0, x)\delta\phi\right] \\ &= \delta\sigma\phi - \lambda\left[\frac{f_{ww}(\lambda, C_{\delta\phi}, x)}{2}(\delta\phi)^2\right] \\ &= \delta\phi\left[\sigma - \frac{\lambda f_{ww}(\lambda, C_{\delta\phi}, x)}{2}\delta\phi\right] \end{aligned}$$

where $C_{\delta\phi} \in (0, \delta\phi)$ comes from Taylor's expansion of f.

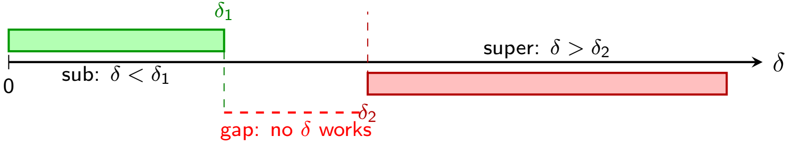
Since $\delta\phi > 0$, sign of I determined by bracket: $\sigma < 0$, $-f_{ww} > 0$

- $\delta < \delta_1 := \dfrac{-2\sigma}{\lambda m^*} \Rightarrow I < 0 \Rightarrow$ **strict subsolution**
- $\delta > \delta_2 := \dfrac{-2\sigma}{\lambda M^* \min_{\overline{\Omega}}\{\phi\}} \Rightarrow I > 0 \Rightarrow$ **strict supersolution**

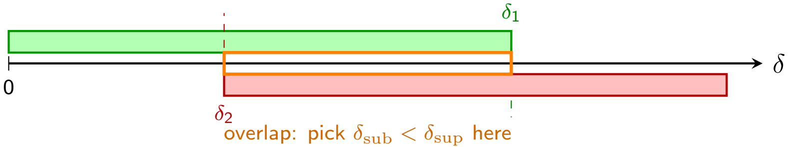
Why the Ordering $\delta_2 < \delta_1$ Matters

Pao's theorem needs $\psi \le Z$ — we must pick a subsolution $\psi = \delta_{\text{sub}}\phi$ *below* a supersolution $Z = \delta_{\text{sup}}\phi$, i.e. $\delta_{\text{sub}} < \delta_{\text{sup}}$. This is possible iff the sub-region and super-region *overlap*.

Bad case: $\delta_2 > \delta_1$ — gap, no valid pair



Good case: $\delta_2 < \delta_1$ — overlap, pair exists



Recall $\delta_1 = \dfrac{-2\sigma}{\lambda m^*}$, $\quad \delta_2 = \dfrac{-2\sigma}{\lambda M^* \min_{\overline{\Omega}}\{\phi\}}$. Then $\delta_2 < \delta_1 \iff m^* < M^* \min_{\overline{\Omega}}\{\phi\}$ — **not guaranteed!**

Existence & Uniqueness

Problem: $\delta_2 < \delta_1$ requires $m^* < M^* \min_\Omega\{\phi\}$ — not guaranteed!
Solution: Use global supersolution $Z \equiv N$ for $N > 1$.

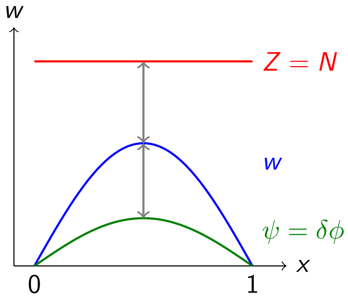
Subsolution: $\psi = \delta\phi$ with small $\delta < \delta_1$

- $-\Delta\psi - \lambda f(\lambda, \psi, x) \leq 0$ in Ω ✓
- BC satisfied ✓

Supersolution: $Z \equiv N$ with $N > 1$

- $-\Delta Z - \lambda f(\lambda, Z, x) = -\lambda f(\lambda, N, x) > 0$ (since $f < 0$ for $w \geq K_{2,\lambda} \leq 1 < N$ by H1(b))
- BC: $\frac{\partial Z}{\partial \eta} + \mu Z = \mu N \geq 0$

Ordering: $\psi = \delta\phi < N = Z$ for small δ



Sub and Super Solution Theorem from C.V. Pao

$\psi \leq Z \Rightarrow \exists$ solution w with $\psi \leq w \leq Z$

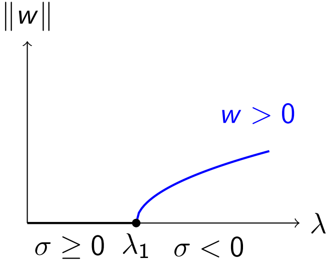
Uniqueness proof is standard using Green's Identity for $w_1 < w_2 < 1$ and Assumption H1

Properties of the Positive Solution

(i) Bounds: $w \in (0,1)$ on Ω (maximum principle + H1(b))
(ii) Bifurcation: $w \to 0^+$ as $\sigma \to 0^-$ [The supersolution we constructed becomes useful here]

By uniqueness: $w \leq \delta_2 \phi = \dfrac{-2\sigma}{\lambda M^* \min_{\overline{\Omega}}\{\phi\}} \phi$

- Numerator: $-2\sigma \to 0^+$ ✓
- $\min_{\Omega}\{\phi\} \not\to 0$ since at $\sigma = 0$, ϕ is Robin eigenfunction with $\phi > 0$ on $\overline{\Omega}$
- $\Rightarrow w \to 0^+$



Global supersolution $Z = N$ only gives $w < N$ — useless for bifurcation!

(iii) Monotonicity: Guaranteed by uniqueness $\mu_1 \leq \mu_2 \Rightarrow w_1 \geq w_2$ (hostile boundary $\Rightarrow$ smaller population) ☐

Minimum Patch Size: Robin Eigenvalue

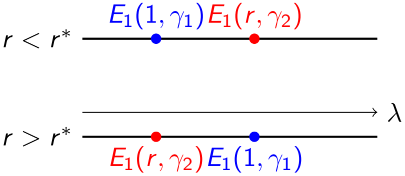
Classical Robin EVP: $-\Delta\phi = ER\phi$ in Ω, $\quad \frac{\partial\phi}{\partial\eta} + \sqrt{\lambda}\gamma\phi = 0$ on $\partial\Omega$

Principal eigenvalue $E_1(R,\gamma)$ R G-D ratio and γ boundary matrix hostility determines **minimum patch size**:

- $\lambda \leq E_1(R,\gamma)$: population dies out
- $\lambda > E_1(R,\gamma)$: population persists

Lemma (Acharya, Bandyopadhyay, Cronin, Goddard, Muthunayake, Shivaji 2023): $\gamma_1 =$ matrix hostility for u, $\gamma_2 =$ matrix hostility for v; $\Rightarrow \exists$ unique $r^*(\gamma_1,\gamma_2)$ such that:

- $r < r^* \Rightarrow E_1(1,\gamma_1) < E_1(r,\gamma_2)$
- $r = r^* \Rightarrow E_1(1,\gamma_1) = E_1(r,\gamma_2)$
- $r > r^* \Rightarrow E_1(r,\gamma_2) < E_1(1,\gamma_1)$



Case A ($r < r^*$): dominant u has smaller min patch size
Case B ($r > r^*$): inferior v has smaller min patch size

Three Eigenvalues

σ_1: **Species u alone**

$$-\Delta\phi - \lambda\phi = \sigma_1\phi, \quad \frac{\partial\phi}{\partial\eta} + \sqrt{\lambda}\gamma_1\phi = 0$$

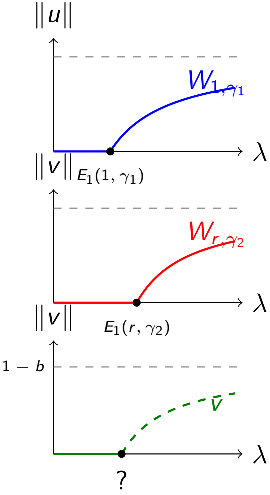
σ_2: **Species v alone**

$$-\Delta\phi - \lambda r\phi = \sigma_2\phi, \quad \frac{\partial\phi}{\partial\eta} + \sqrt{\lambda}\gamma_2\phi = 0$$

σ_3: **v with u present**

$$-\Delta\phi - \lambda r(1 - bW_{1,\gamma_1})\phi = \sigma_3\phi$$

$$\frac{\partial\phi}{\partial\eta} + \sqrt{\lambda}\gamma_2 h(W_{1,\gamma_1}, \beta)\phi = 0$$



σ_1, σ_2: constant coefficients ⇒ explicit threshold
σ_3: spatially varying ⇒ depend on x, λ, β, b. No simple threshold!

Connecting Single Species Theorem to Thresholds

Single Species Theorem: $\sigma \geq 0 \Rightarrow$ extinction $\quad \sigma < 0 \Rightarrow$ persistence
For σ_1: Eigenvalue problem with constant coefficients

$$-\Delta\phi - \lambda\phi = \sigma_1\phi \quad \Leftrightarrow \quad -\Delta\phi = (\sigma_1 + \lambda)\phi$$

Comparing with Robin problem $-\Delta\phi = E \cdot R \cdot \phi$ with $R = 1$:

$$\sigma_1 + \lambda = E_1(1, \gamma_1) \quad \Rightarrow \quad \boxed{\sigma_1 = E_1(1, \gamma_1) - \lambda}$$

For σ_2: Same argument with $R = r$:

$$\boxed{\sigma_2 = E_1(r, \gamma_2) - \lambda}$$

For σ_3: Coefficients $r(1 - bW_{1,\gamma_1}(x))$ and $h(W_{1,\gamma_1}(x), \beta)$ depend on x!
non constant coefficients give trouble to characterize the threshold with principle EV $\sigma_3 \neq E_1(\cdot, \cdot) - \lambda$?????

σ_1, σ_2: sign determined by comparing λ with E_1
σ_3: need following Lemma (Green's identity) to determine sign

Lemma 1: The Tool to determine the sign of σ_3

Lemma 1: Given $A \in [0,1]$ and $\lambda > \max\{E_1(r, \gamma_2 h(A,\beta)), E_1(1,\gamma_1)\}$:

$$\sigma_3 \int_\Omega W\phi_3 \, dx = \lambda r \int_\Omega W\phi_3(bW_{1,\gamma_1} - W)\, dx + \sqrt{\lambda}\gamma_2 \int_{\partial\Omega} W\phi_3[h(W_{1,\gamma_1},\beta) - h(A,\beta)]\, ds$$

Proof: Green's Second Identity with W and ϕ_3:

$$\int_\Omega (-\Delta W)\phi_3 + (\Delta\phi_3) W \, dx = \int_{\partial\Omega} \left(-\frac{\partial W}{\partial\eta}\phi_3 + \frac{\partial\phi_3}{\partial\eta} W \right) ds$$

Left side: Substitute $-\Delta W = \lambda r W(1-W)$ and $-\Delta\phi_3 = \sigma_3\phi_3 + \lambda r(1 - bW_{1,\gamma_1})\phi_3$:

$$\int_\Omega \lambda r W(1-W)\phi_3 - \sigma_3 W\phi_3 - \lambda r(1 - bW_{1,\gamma_1})W\phi_3 \, dx$$

$$= \lambda r \int_\Omega W\phi_3(bW_{1,\gamma_1} - W)\, dx - \sigma_3 \int_\Omega W\phi_3 \, dx$$

Right side: Use BCs $\frac{\partial W}{\partial\eta} = -\sqrt{\lambda}\gamma_2 h(A,\beta) W$ and $\frac{\partial\phi_3}{\partial\eta} = -\sqrt{\lambda}\gamma_2 h(W_{1,\gamma_1},\beta)\phi_3$:

$$\int_{\partial\Omega} \sqrt{\lambda}\gamma_2 h(A,\beta) W\phi_3 - \sqrt{\lambda}\gamma_2 h(W_{1,\gamma_1},\beta) W\phi_3 \, ds$$

$$= -\sqrt{\lambda}\gamma_2 \int_{\partial\Omega} W\phi_3[h(W_{1,\gamma_1},\beta) - h(A,\beta)]\, ds$$

LHS= RHS and rearrange.

Using Lemma 1: Strategic Choice of A

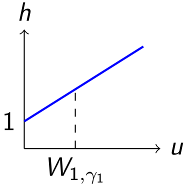
$$\sigma_3 \underbrace{\int_\Omega W\phi_3\,dx}_{>0} = \underbrace{\lambda r \int_\Omega W\phi_3(bW_{1,\gamma_1} - W)\,dx}_{\text{interior}} + \underbrace{\sqrt{\lambda}\gamma_2 \int_{\partial\Omega} W\phi_3[h(W_{1,\gamma_1},\beta) - h(A,\beta)]\,ds}_{\text{boundary}}$$

$$\text{sign}(\sigma_3) = \text{sign}(\text{RHS}) \quad \Rightarrow \quad \begin{cases} \text{RHS} > 0 \Rightarrow \sigma_3 > 0 \Rightarrow \text{nonexistence} \\ \text{RHS} < 0 \Rightarrow \sigma_3 < 0 \Rightarrow \text{existence} \end{cases}$$

Interior term: sign depends on $bW_{1,\gamma_1} \gtrless W$ (controlled by b)

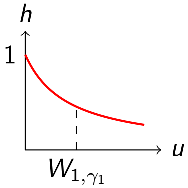
Boundary term: sign depends on h and choice of A

+DDE: $h(u,\beta) = \beta u + 1$ (increasing)



$A = 0$: $h(W_{1,\gamma_1},\beta) - h(0,\beta) > 0$
$A = 1$: $h(W_{1,\gamma_1},\beta) - h(1,\beta) < 0$

-DDE: $h(u,\beta) = \frac{1}{\beta u+1}$ (decreasing)



$A = 0$: $h(W_{1,\gamma_1},\beta) - h(0,\beta) < 0$
$A = 1$: $h(W_{1,\gamma_1},\beta) - h(1,\beta) > 0$

Same A, opposite boundary signs ⇒ +DDE and -DDE behave differently!

Lemma 2: Global Nonexistence

Problem: Lemma 1 gives nonexistence on **bounded** λ-intervals when $\sigma \geq 0$.
For some parameters, need nonexistence for **all** $\lambda > E_1(1, \gamma_1)$.

Lemma 2

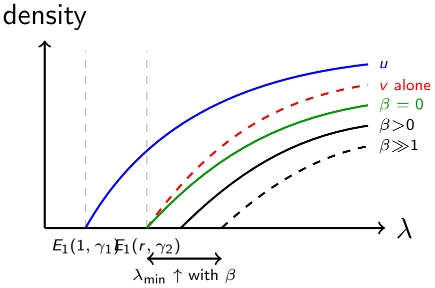
If (u, v) is a positive solution, then:

$$\lambda \int_\Omega uv[(1-r) + (rb-1)u + rv]\, dx = \sqrt{\lambda} \int_{\partial\Omega} uv[\gamma_1 - \gamma_2 h(u, \beta)]\, ds$$

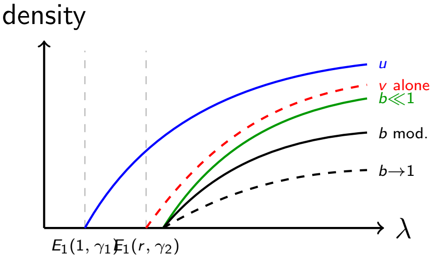
Proof: Green's Second Identity with u and v.
(ref., Acharya, Bandyopadhyay, Cronin, Goddard, Muthunayake, Shivaji 2023)
Application: If $r \leq 1$, $\gamma_1 \leq \gamma_2$ (one strict), and $b \geq \frac{1}{r}$:

- LHS: $(1-r) + (rb-1)u + rv \geq 0$ (since $r \leq 1$, $rb \geq 1$, $u, v > 0$)
- RHS: $\gamma_1 - \gamma_2 h(u, \beta) < \gamma_1 - \gamma_2 < 0$ (since $h \geq 1$, $\gamma_1 < \gamma_2$)
- LHS ≥ 0, RHS < 0 $\Rightarrow$ contradiction $\Rightarrow$ no positive solution for any λ!

+DDE Case A: $E_1(1,\gamma_1) < E_1(r,\gamma_2)$



vary β, $b = 0.6$ fixed

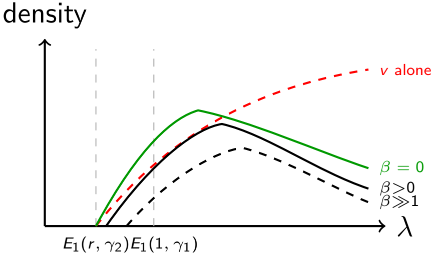


vary b, $\beta > 0$ fixed

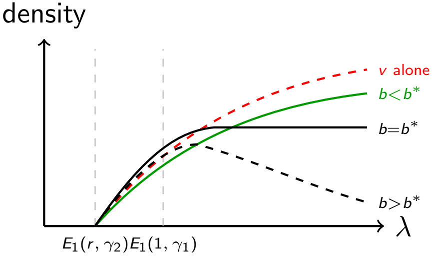
Key observations:

- Larger $\beta \Rightarrow$ higher $\lambda_{\min}$ for v (boundary more hostile)
- v density always *below* v-alone curve (competition hurts)
- Positive DAR as $\lambda \to \infty$ for $b < 1$

+DDE Case B: $E_1(1,\gamma_1) > E_1(r,\gamma_2)$



vary β, $b > b^*$ fixed



vary b, $\beta > 0$ fixed

Key observations: Critical $b^* = 1 - \int_\Omega W_{r,\gamma_2}(E_1(1,\gamma_1), x)\,dx$

- $b < b^*$: positive DAR • $b = b^*$: neutral • $b > b^*$: **hump-shaped**
- Larger β amplifies the hump

Lemma 3: Density Bounds for $-$DDE Case

Setting: $-$DDE (refuge-seeking): $h(u,\beta) = \dfrac{1}{\beta u + 1}$ (decreasing in u)

Key observation: As $\beta \to \infty$, $h \to 0 \Rightarrow$ boundary becomes **less** hostile!

Lemma 3 (Main Results)

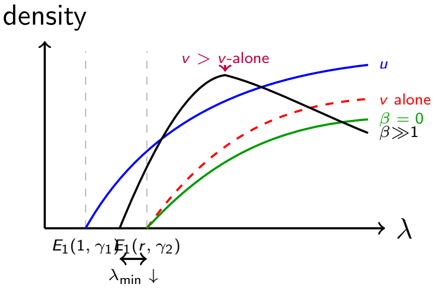
Fix $r, \gamma_1, \gamma_2 > 0$, $b \geq 0$, and assume v is a positive solution. Then:

1. If $b < 1$ and $\beta \gg 1$: $\displaystyle\int_\Omega v\,dx > 1 - b$
2. If $b \ll 1$: $\displaystyle\int_\Omega v\,dx > \int_\Omega W_{r,\gamma_2}\,dx$ (v beats v-alone!)
3. If $\lambda \approx E_1(1,\gamma_1)$: $\displaystyle\int_\Omega v\,dx > \int_\Omega W_{r,\gamma_2}\,dx$

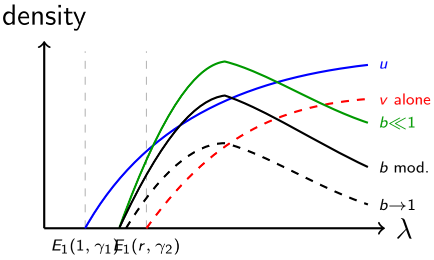
Proof idea:

- Auxiliary problem: $-\Delta Z = \lambda r Z(1-Z)$ with BC using $h(u,\beta)$ but $b = 0$
- As $\beta \to \infty$: $Z \to 1$ uniformly (boundary hostility $\to 0$)
- Subsolution $\psi = (1-\theta)Z \Rightarrow$ sandwich: $(1-\theta)Z \leq v \leq Z$

$-$DDE Case A: $E_1(1,\gamma_1) < E_1(r,\gamma_2)$



vary β, $b = 0.6$ fixed

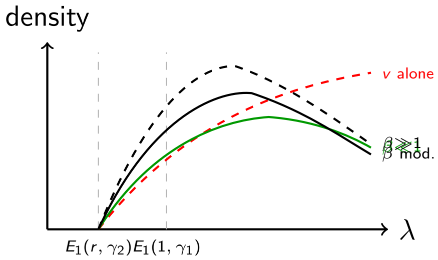


vary b, $\beta \gg 1$ fixed

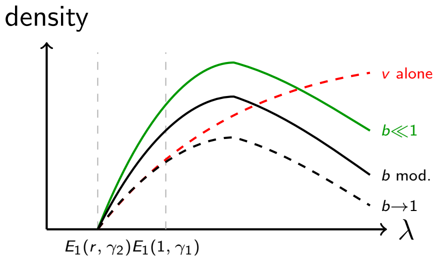
Key observations:

- Larger $\beta \Rightarrow$ *lower* $\lambda_{\min}$ (boundary less hostile!)
- v can *exceed* v-alone (refuge benefit > competition cost)
- Hump-shaped DAR: v rises, peaks, then falls toward $1 - b$

$-$DDE Case B: $E_1(1,\gamma_1) > E_1(r,\gamma_2)$



vary β, $b < 1$ fixed



vary b, $\beta \gg 1$ fixed

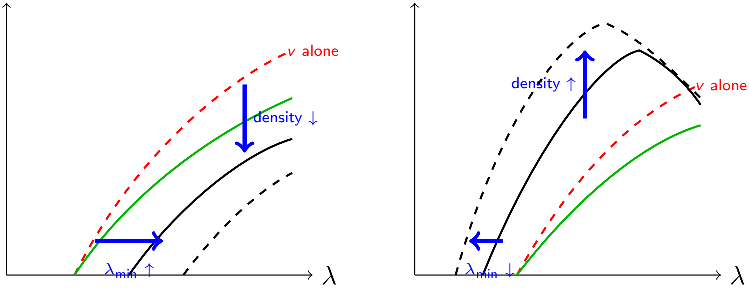
Key observations:

- Small β: like $+$DDE (below v-alone); moderate β: $v >$ v-alone briefly; large β: strong refuge
- $\int_\Omega v\,dx > \int_\Omega W_{r,\gamma_2}\,dx$ in some λ-range (Lemma 3)

Comparison: $+$DDE vs $-$DDE

+DDE: $h = \beta u + 1$

$-$**DDE:** $h = \frac{1}{\beta u + 1}$



	+DDE (dominant-avoiding)	$-$**DDE** (refuge-seeking)
Boundary effect	more hostile	less hostile
$\lambda_{\min}$	increases with β	decreases with β
Density vs *v*-alone	always below	can exceed!
Biological meaning	avoid dominant → leave patch	seek refuge → stay in patch

Future Directions (1/2)

(1) Strong Allee effect in v.

$$\begin{aligned} &\text{in } \Omega: && -\Delta u = \lambda u(1-u), && -\Delta v = \lambda r\, v\,(v-a)\,(1-v-bu), \quad a \in (0,1). \\ &\text{on } \partial\Omega: && \tfrac{\partial u}{\partial \eta} + \sqrt{\lambda}\,\gamma_1\, u = 0, && \tfrac{\partial v}{\partial \eta} + \sqrt{\lambda}\,\gamma_2\, h(u,\beta)\, v = 0. \end{aligned}$$

Extends Cronin, Fonseka, Goddard II, Leonard, Shivaji (*MBE*, 2019) — single-species DDE + weak Allee — to two species.

(2) Heterogeneous matrix hostility.

$$\begin{aligned} &\text{in } \Omega: && -\Delta u = \lambda u(1-u), && -\Delta v = \lambda r\, v\big(1-v-b(x)\,u\big). \\ &\text{on } \partial\Omega: && \tfrac{\partial u}{\partial \eta} + \sqrt{\lambda}\,\gamma_1(x)\, u = 0, && \tfrac{\partial v}{\partial \eta} + \sqrt{\lambda}\,\gamma_2(x)\, h(u,\beta)\, v = 0, \end{aligned}$$

$$\gamma_i : \partial\Omega \to [0,\infty), \qquad b : \Omega \to [0,\infty).$$

Motivated by Cantrell & Cosner, *Spatial Ecology via RD Equations* (Wiley, 2003) — spatially varying Robin data.

Future Directions (2/2)

(3) Non-monotone (U-shaped) h.

$$\begin{aligned} &\text{in } \Omega: && -\Delta u = \lambda u(1-u), && -\Delta v = \lambda r\, v\,(1-v-bu). \\ &\text{on } \partial\Omega: && \tfrac{\partial u}{\partial \eta} + \sqrt{\lambda}\,\gamma_1\, u = 0, && \tfrac{\partial v}{\partial \eta} + \sqrt{\lambda}\,\gamma_2\, h(u,\beta)\, v = 0, \end{aligned}$$

$$h(u,\beta) = 1 + \beta\,(u-u_0)^2, \qquad u_0 \in (0,1).$$

Extends Goddard II, Morris, Payne, Shivaji (*TMNA*, 2019) — single-species U-shaped DDE — to two species.

(4) Fully coupled symmetric competition.

$$\begin{aligned} &\text{in } \Omega: && -\Delta u = \lambda\, u\,(1-u-c\,v), && -\Delta v = \lambda r\, v\,(1-v-b\,u). \\ &\text{on } \partial\Omega: && \tfrac{\partial u}{\partial \eta} + \sqrt{\lambda}\,\gamma_1\, h_1(v,\beta_1)\, u = 0, && \\ & && \tfrac{\partial v}{\partial \eta} + \sqrt{\lambda}\,\gamma_2\, h_2(u,\beta_2)\, v = 0. && \end{aligned}$$

Symmetric extension of Acharya, Bandyopadhyay, Cronin, Goddard II, Muthunayake, Shivaji (*NARWA*, 2023) and the present work.

Thank You!

Questions?

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