# Nonlinear BVP on Discrete Time-Scale

S. Bandyopadhyay K. Byassee C. Lynch

Department of Mathematics & Statistics, University of Tennessee at Martin

## Boundary Value Problem

$T \in \mathbb{N}$ be fixed.
Discrete Interval
$\mathbb{T} := [1,\ T+1] := \{1,\ 2,\ \dots,\ T,\ T+1\}$.
Consider the difference equation:

$$\begin{cases} \Delta^2 u(t-1) + \tilde{h}(t,\ u(t),\ \Delta u(t-1)) = 0;\ t \in \mathbb{T} \\ \Delta u(0) = 0 \\ u(T+2) = g(T+2) \end{cases} \tag{BVP}$$

## Upper solution $\beta$

$\beta\colon [0,\ T+2] \to \mathbb{R}$ is called an upper function of (BVP) if

$$\begin{cases} \Delta^2 \beta(t-1) + h(t,\ \beta(t),\ \Delta\beta(t-1)) \le 0, \text{ for } t \in [1,\ T+1] \\ \Delta\beta(0) \le 0 \\ \beta(T+2) \ge g(T+2) \end{cases}$$

## Lower solution $\alpha$

$\alpha\colon [0,\ T+2] \to \mathbb{R}$ is a lower function of (BVP) if

$$\begin{cases} \Delta^2 \alpha(t-1) + h(t,\ \alpha(t),\ \Delta\alpha(t-1)) \ge 0, \text{ for } t \in [1,\ T+1] \\ \Delta\alpha(0) \ge 0 \\ \alpha(T+2) \le g(T+2) \end{cases}$$

## Theorem

✓ $\alpha$ and $\beta$ be a lower and an upper function of (BVP).
✓ $\alpha \le \beta$ on $[1,\ T+1]$.
✓ $h$ be continuous on $[1,\ T+1] \times \mathbb{R}^2$.
✓ $h$ is non-increasing in its third variable.

Then (BVP) has a solution $u(t)$ satisfying:

$$\alpha(t) \le u(t) \le \beta(t), \quad t \in [0,\ T+2]$$

## Define $\sigma$ and $\tilde{h}$

For $t \in [1,\ T+1]$, $x,\ z \in \mathbb{R}$, define functions:

$$\sigma(t,\ z) = \begin{cases} \beta(t-1), & z > \beta(t-1) \\ z, & \alpha(t-1) \le z \le \beta(t-1) \\ \alpha(t-1), & z < \alpha(t-1) \end{cases}$$

$$\tilde{h}(t,\ x,\ x-z) = \begin{cases} h(t,\ \beta(t),\ \beta(t) - \sigma(t,\ z)) - \dfrac{x - \beta(t)}{x - \beta(t) + 1}, & x > \beta(t) \\ h(t,\ x,\ x - \sigma(t,\ z)), & \alpha(t) \le x \le \beta(t) \\ h(t,\ \alpha(t),\ \alpha(t) - \sigma(t,\ z)) + \dfrac{\alpha(t) - x}{\alpha(t) - x + 1}, & x < \alpha(t) \end{cases}$$

## Brower's fixed point theorem

- Any continuous mapping from the unit ball to itself has a fixed point.

## Define $\mathcal{T}$

✓ $E := \{u\colon [0,\ T+2] \to \mathbb{R},\ \Delta u(0) = 0,\ u(T+2) = g(T+2)\}$
✓

$$\|u\| = \max\{u(t)\colon t \in [1,\ T+1]\}$$

$E$ is a Banach space with $\dim E = T+1$.
Define $\mathcal{T}\colon E \to E$ by

$$(\mathcal{T}u)(t) = \sum_{s=t}^{T+1} \left( \sum_{i=1}^{s} \tilde{h}(i,\ u(i),\ \Delta u(i-1)) \right),\ t \in [0,\ T+2] \tag{1}$$

Hence, $\mathcal{T}$ is a continuous operator. Moreover (1) implies that if

$$r \ge \sum_{s=1}^{T+1} sM$$

Then $\mathcal{T}(\overline{B(r)}) \subseteq \overline{B(r)}$, where $B(r) = \{u \in E\colon \|u\| < r\}$. Therefore, by the Brouwer Fixed Point Theorem $\exists u \in \overline{B(r)}$ such that $u = \mathcal{T}u$.

## $u$ is a fixed point of $\mathcal{T}$ iff $u$ is a solution

$(\Longrightarrow)$
Assume $u = \mathcal{T}u$.
Then

$$\Delta\Delta u(t-1) = \Delta u(t) - \Delta u(t-1)$$
$$= -\tilde{h}(t,\ u(t),\ \Delta u(t-1)) \text{ for } t \in [1,\ T+1].$$

$(\Longleftarrow)$
Assume $u$ solves (BVP). Then $u \in E$ and $\Delta u(0) = 0$. We proceed with a proof by induction to show.

$$u(t) = g(T+2) + \sum_{s=t}^{T+1} \sum_{i=1}^{s} \tilde{h}(i,\ u(i),\ \Delta u(i-1)), \quad t \in [0,\ T+2] \tag{2}$$

## Final Step:

$$\alpha(t) \le u(t) \le \beta(t)$$

To show, $u$ of (BVP) satisfy

$$\alpha(t) \le u(t) \le \beta(t), \quad t \in [0,\ T+2]$$

- $v(t) = u(t) - \beta(t)$
- assume for a proof by contradiction that $\max\{v(t)\colon t \in [0,\ T+2]\} = v(\ell) > 0$.
- This implies $\Delta\Delta u(\ell-1) \le \Delta\Delta\beta(\ell-1)$.
- Next show that $\Delta\Delta u(\ell-1) - \Delta\Delta\beta(\ell-1) > 0$.

## Selected References


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