Positive Solutions of Elliptic Systems with Superlinear Boundary Nonlinearities: Bifurcation from Zero and Infinity

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The Problem

We study the elliptic system with nonlinear boundary conditions:

$$
\begin{aligned}
-\Delta u_1 + u_1 &= 0 \quad \text{in } \Omega \\
\frac{\partial u_1}{\partial \eta} &= \lambda f_1(u_2) \quad \text{on } \partial\Omega \\
-\Delta u_2 + u_2 &= 0 \quad \text{in } \Omega \\
\frac{\partial u_2}{\partial \eta} &= \lambda f_2(u_1) \quad \text{on } \partial\Omega
\end{aligned} \tag{*}
$$

Assumptions:

- $\Omega \subset \mathbb{R}^N$ $(N > 2)$ bounded with $C^{2,\alpha}$ boundary
- $f_i : [0, \infty) \to [0, \infty)$ Hölder continuous
- Superlinear growth at infinity: $\lim_{s\to\infty} \frac{f_i(s)}{s^{p_j}} = b_j > 0;\ i \neq j$
- Subcritical: $1 < p_1, p_2 \leq \frac{N}{N-2}$ but not both equal to $\frac{N}{N-2}$

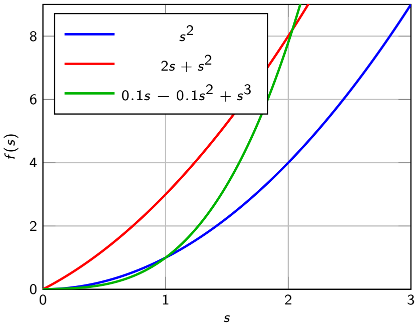
Superlinear Subcritical Growth

The function f satisfies the growth condition:

Hypothesis $(\mathcal{H}_\infty)$

$\lim_{s\to+\infty} \frac{f(s)}{s^p} = b > 0$ with

$$\begin{cases} 1 < p < \frac{N}{N-2} & \text{if } N \geq 3 \\ p > 1 & \text{if } N = 2 \end{cases}$$



Examples:

- $f(s) = s^2$ (pure superlinear)
- $f(s) = 2s + s^2$ (mixed)
- $f(s) = 0.1s - 0.1s^2 + s^3$ (cubic)

$f(u) = u^p +$ lower order terms; When u is small $f(u) = u^p$.

Weak Formulation and Solution Space

$(\lambda, u_1, u_2) \in (0,\infty)(H^1(\Omega))^2$ is a weak solution if:

$$\int_\Omega \nabla u_1 \nabla \psi_1 + \int_\Omega u_1\psi_1 = \lambda \int_{\partial\Omega} f_1(u_2)\psi_1$$
$$\int_\Omega \nabla u_2 \nabla \psi_2 + \int_\Omega u_2\psi_2 = \lambda \int_{\partial\Omega} f_2(u_1)\psi_2$$

for all $\psi_1, \psi_2 \in H^1(\Omega)$.

Regularity: Weak solutions are $C^{2,\alpha}(\overline{\Omega})$.

Solution set:

$$\Sigma := \{(\lambda, (u_1, u_2)) \in (0,\infty) \times (C(\overline{\Omega}))^2 : \text{weak solution of } (*)\}$$

with norm $\|(u_1, u_2)\| := \|u_1\|_{C(\overline{\Omega})} + \|u_2\|_{C(\overline{\Omega})}$.

Well-definedness of Weak Formulation

Goal: Show that under (H_∞), the integrals $\int_{\partial\Omega} f_i(u_j)\psi$ are well-defined for $u_1, u_2 \in H^1(\Omega)$ and $\psi \in H^1(\Omega)$.

Hypothesis reminder:

$$\lim_{s\to\infty} \frac{f_1(s)}{s^{p_2}} = b_2, \quad \lim_{s\to\infty} \frac{f_2(s)}{s^{p_1}} = b_1$$

with $1 < p_1, p_2 \le \frac{N}{N-2}$ but not both equal to $\frac{N}{N-2}$.

Step 1: Growth estimate from above: For large s, $f_i(s) \le C(1 + s^{p_j})$ for some $C > 0$.

Step 2: Sobolev trace theorem: $H^1(\Omega) \hookrightarrow L^{2^*}(\partial\Omega)$ where $2^* = \frac{2(N-1)}{N-2}$.

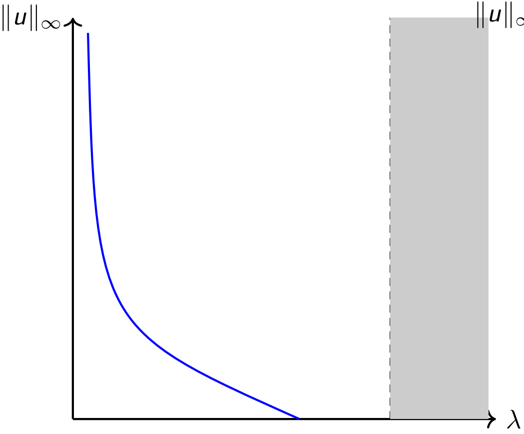
Step 3: Check Hölder inequality conditions: - Need $f_i(u_j) \in L^q(\partial\Omega)$ and $\psi \in L^{q'}(\partial\Omega)$ with $\frac{1}{q} + \frac{1}{q'} = 1$ - Since $u_j \in L^{2^*}(\partial\Omega)$ and $f_i(u_j) \le C(1 + |u_j|^{p_j})$:

$$f_i(u_j) \in L^{2^*/p_j}(\partial\Omega)$$

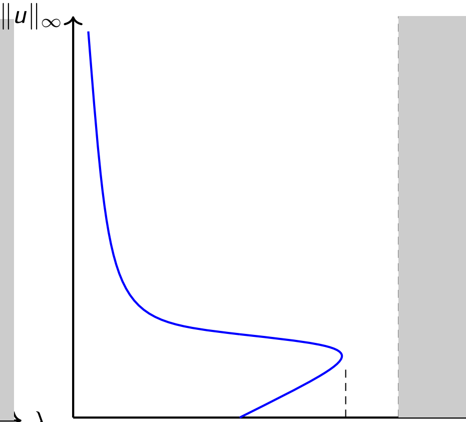
Step 4: Verification: $\frac{2(N-1)}{p_j(N-2)} \ge 1$ since $p_j \le \frac{N}{N-2}$, so $q = \frac{2^*}{p_j} \ge 1$.
The conjugate exponent $q' = \frac{2^*}{2^*-p_j}$ satisfies $q' \le 2^*$, so $\psi \in L^{q'}(\partial\Omega)$.

Conclusion: Hölder inequality ensures $\int_{\partial\Omega} f_i(u_j)\psi < \infty$.

Expected Bifurcation Diagrams



to the left



to the right (multiplicity)

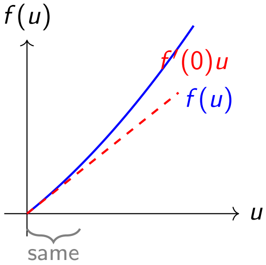
Principal Steklov Eigenvalue Controls Everything

Our nonlinear problem:

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \eta} = \lambda f(u) & \text{on } \partial\Omega \end{cases}$$

Linearize near $u = 0$:

$$\begin{cases} -\Delta \phi + \phi = 0 & \text{in } \Omega \\ \dfrac{\partial \phi}{\partial \eta} = \mu f'(0)\phi & \text{on } \partial\Omega \end{cases}$$



Small u: $f(u) \approx f'(0)u$
Nonlinear $\rightarrow$ linear $\rightarrow$ Steklov EVP

μ_1 characterizes where positive solution starts to exists.
$\phi_1 > 0$ denotes the first eigenfunction corresponding to μ_1.

The Trace Operator

Functions in $H^1(\Omega)$ are defined **almost everywhere** — they have no well-defined values on $\partial\Omega$, which has measure zero.

The Trace Operator: There exists a bounded linear operator

$$\Gamma : H^1(\Omega) \longrightarrow L^2(\partial\Omega)$$

such that $\Gamma u = u|_{\partial\Omega}$ whenever $u \in C(\overline{\Omega})$.

Key properties:

- **Extension of restriction:** For smooth functions, Γu is just the boundary value.
- **Continuity:** $\|\Gamma u\|_{L^2(\partial\Omega)} \leq C\|u\|_{H^1(\Omega)}$
- **Sobolev trace embedding:** $\Gamma : H^1(\Omega) \hookrightarrow L^{2^*}(\partial\Omega), \quad 2^* = \dfrac{2(N-1)}{N-2}$
- **Compact** for subcritical exponents: $\Gamma : H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$ for $q < 2^*$

Our boundary nonlinearities $f_i(u_j)|_{\partial\Omega}$ only make sense through the trace. The integrals $\int_{\partial\Omega} f_i(u_j)\psi$ in the weak formulation are well-defined precisely because $\Gamma u_j \in L^{2^*}(\partial\Omega)$.

Theorem 1 (Local Bifurcation from Infinity)

There exists $\widetilde{\lambda} > 0$ such that for all $\lambda \in (0, \widetilde{\lambda}]$, system $(*)$ has a positive weak solution $(\lambda, (u_1, u_2))$ with

$$\|(u_1, u_2)\|_{(C(\overline{\Omega}))^2} \to \infty \quad \text{as } \lambda \to 0^+.$$

Moreover, there exists a connected component $\mathscr{C}^+ \subset \Sigma$ of positive weak solutions bifurcating from infinity at $\lambda = 0$, whose projection onto the parameter space is $(0, \widetilde{\lambda}]$.

Proof Strategy:

1. **(Rescaling):** Set $w_i = \lambda^{\theta_i} u_i$ to reduce to a pure-power limiting system.
2. **(Degree Theory):** Leray–Schauder degree + excision $\Rightarrow$ existence.

Proposition (Bonder-Rossi 2003)

Suppose (H_∞) holds for $i = 1, 2$. Then there exists $M > 0$ such that every nonnegative solution (u_1, u_2) of $(*)$ satisfies

$$\|(u_1, u_2)\|_{(C(\overline{\Omega}))^2} \leq M,$$

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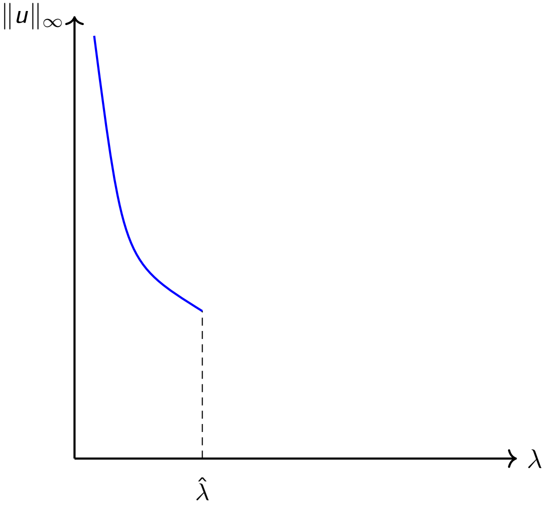
Expected Bifurcation from infinity at $\lambda = 0$



Proof Step 1: Rescaling Method

Transform the original system to isolate the essential asymptotic behavior as $\lambda \to 0^+$

Original Problem:

$$\begin{cases} -\Delta u_1 + u_1 = 0 \quad \text{in } \Omega \\ \dfrac{\partial u_1}{\partial \eta} = \lambda f_1(u_2) \quad \text{on } \partial\Omega \\ -\Delta u_2 + u_2 = 0 \quad \text{in } \Omega \\ \dfrac{\partial u_2}{\partial \eta} = \lambda f_2(u_1) \quad \text{on } \partial\Omega \end{cases}$$

Challenge: As $\lambda \to 0^+$, solutions blow up: $\|(u_1, u_2)\|_\infty \to +\infty$

Rescaled Problem: Set $w_i = \lambda^{\theta_i} u_i$, where $\theta_1, \theta_2 > 0$ satisfy

$$1 + \theta_2 - \theta_1 p_1 = 0, \quad 1 + \theta_1 - \theta_2 p_2 = 0,$$

then (w_1, w_2) solves:

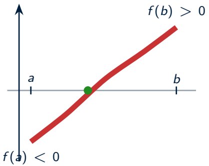
$$\begin{cases} -\Delta w_1 + w_1 = 0 \quad \text{in } \Omega \\ \dfrac{\partial w_1}{\partial \eta} = \widetilde{f}_1(\lambda, w_2) \quad \text{on } \partial\Omega \\ -\Delta w_2 + w_2 = 0 \quad \text{in } \Omega \\ \dfrac{\partial w_2}{\partial \eta} = \widetilde{f}_2(\lambda, w_1) \quad \text{on } \partial\Omega \end{cases}$$

where $\widetilde{f}_1(\lambda, w_2) \to b_2 w_2^{p_2}$ and $\widetilde{f}_2(\lambda, w_1) \to b_1 w_1^{p_1}$ as $\lambda \to 0^+$

The rescaling extracts the **pure power nonlinearities** $b_2 w_2^{p_2}$ and $b_1 w_1^{p_1}$ from the general functions f_1, f_2 via condition (H_∞). This transforms an unbounded coupled problem into a bounded one with a well-defined limit.

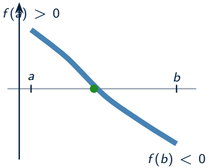
Step 2: How Do We Prove Solutions Exist?

Start with something familiar: If $f(a) < 0$ and $f(b) > 0$, then f has a zero in (a, b).



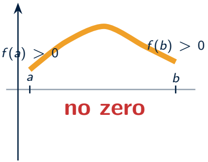
deg $= +1$

Crosses **upward**



deg $= -1$

Crosses **downward**



deg $= 0$

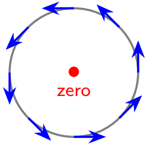
Same sign, **might not cross**

Key insight: $\deg \neq 0 \iff$ sign change at boundary $\implies$ **zero must exist inside**

Leray–Schauder Degree: same idea in higher dimensions. In $\mathbb{R}^N$: does the function **wind around** zero on $\partial\Omega$? If yes $(\deg \neq 0) \Rightarrow$ solution **trapped inside**.

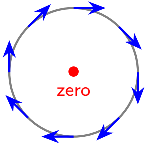
Understanding Degree

deg $= +1$



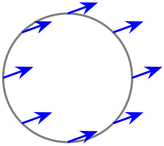
Winds once CCW
Zero trapped!

deg $= -1$



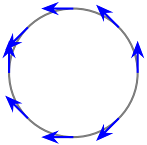
Winds once CW
Zero trapped!

deg $= 0$



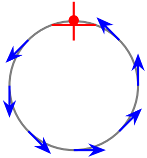
No winding (wobble)
Can't conclude zero

deg $= 0$



Winds cancel
Can't conclude zero

Undefined!



Zero on boundary
Degree undefined

Key insight: deg $\neq 0 \implies$ zero must exist inside (it's "lassoed")

Proof Step 2: PDE to Operator Equation

Track only boundary values — reformulate as a fixed point problem on $\partial\Omega$.

The Round Trip on $\partial\Omega$:

1. $\widetilde{F}$: boundary values $(z_1, z_2) \mapsto$ nonlinear flux
2. T: flux $\mapsto$ solve interior $(-\Delta w_i + w_i = 0)$
3. Γ: bring solution back to $\partial\Omega$

$\widetilde{G} = S \circ \widetilde{F}, \quad S := \Gamma \circ T$ **(compact)**

Operator Chain:

$$\widetilde{F}(\lambda, (z_1, z_2)) := \begin{pmatrix} \widetilde{f}_1(\lambda, z_2) \\ \widetilde{f}_2(\lambda, z_1) \end{pmatrix}$$

$$S := \Gamma \circ T \; : \; L^q(\partial\Omega) \to L^r(\partial\Omega)$$

$$\widetilde{G}(\lambda, \cdot) := S \circ \widetilde{F}(\lambda, \cdot)$$

$$\widetilde{G} : [0, \infty) \times (C(\partial\Omega))^2 \to (C(\partial\Omega))^2$$

Equivalence

$\widetilde{G}\big(\lambda, (\Gamma u_1, \Gamma u_2)\big) = (\Gamma u_1, \Gamma u_2) \iff (\lambda, (u_1, u_2))$ is a positive weak solution of $(*)$

A fixed point of $\widetilde{G}$ means the boundary values you started with are exactly the ones you got back $\Rightarrow$ Solution of our PDE!

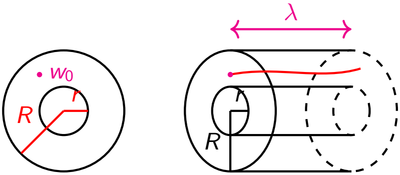
Degree Theory on the Limiting System

Small ball B_r:

- Limiting system has no solution on ∂B_r
- $\deg(I - \widetilde{G}(0,\cdot), B_R(0) \setminus \overline{B_r}(0), 0) = -1$

Large ball B_R:

- A priori bound (Prop. 2.3) $\Rightarrow$ all solutions of $(*)$ satisfy $\|(u_1, u_2)\|_{(C(\bar{\Omega}))^2} \leq M$
- Take $R > M \Rightarrow$ no fixed points of $\widetilde{G}$ on ∂B_R

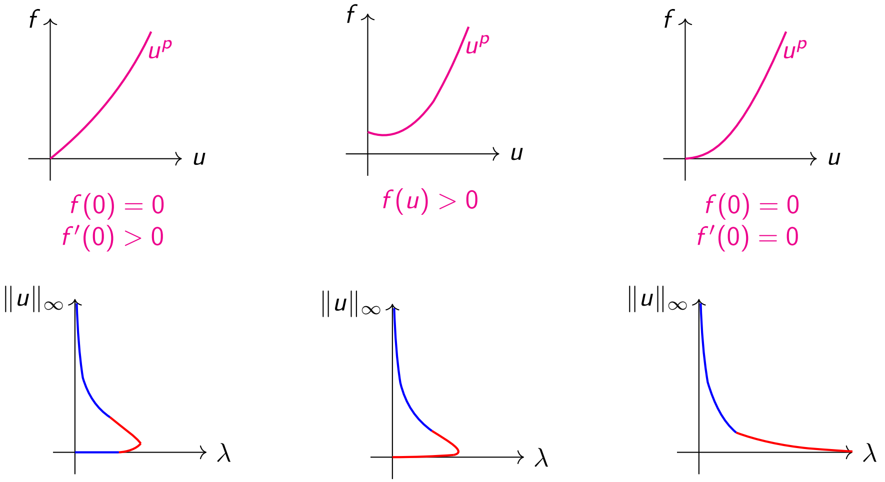


Step 3: Excision $\deg(I - \widetilde{G}(0,\cdot), B_R(0) \setminus \overline{B_r}(0), 0) = -1 \neq 0 \quad \Rightarrow$ solution (w_1, w_2) exists!

Step 4: Homotopy in λ Use $\lambda \geq 0$ as homotopy parameter to transfer existence back to $(*)$.

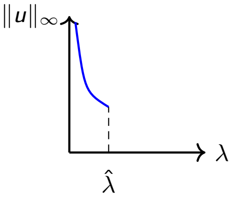
A. Ambrosetti, D. Arcoya, and B. Buffoni. Positive solutions for some semi-positone problems via bifurcation theory. *Differential Integral Equations, 7(3-4):655–663, 1994.*

What Degree Theory Gives us? — Local Bifurcation from ∞

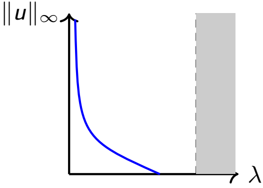


From Local to Global Bifurcation

So far: Bifurcation from infinity at $\lambda = 0$



But we want the full picture.



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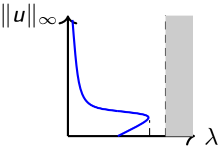


to the right (multiplicity)

Additional Hypotheses for Global Bifurcation

For bifurcation from trivial solution, we impose:

Hypothesis (H_0)

For $i = 1, 2$:

- $f_i(0) = 0,\ f_i'(0) > 0$
- There exist $\nu_i > 1$ such that

$$f_i(s) = f_i'(0)s + R_i(s) \quad \text{for } s \geq 0$$

where $R_i(s) = O(s^{\nu_i})$ as $s \to 0$

Growth Condition

$$f_i(s) \geq Ks \quad \text{for all } s \geq 0, \text{ some } K > 0$$

This ensures $(0, 0)$ is a solution and allows linearization analysis.

Theorem 2: Global Bifurcation Result

Under hypotheses (H_0) and (H_∞), with $f_i(s) \geq Ks$ for all $s \geq 0$, $i = 1, 2$:

Part 1: Existence of Global Branch

There exists a connected component $\mathscr{C}^+ \subset \Sigma$ of positive weak solutions emanating from the trivial solution at $(\mu_0, (0, 0))$, where

$$\mu_0 := \frac{\mu_1}{\sqrt{f_1'(0) f_2'(0)}}$$

Part 2: Behavior of $\mathscr{C}^+$

For any $(\lambda, (u_1, u_2)) \in \mathscr{C}^+$:

$$\|(u_1, u_2)\| \to 0 \text{ as } \lambda \to \mu_0 \qquad \text{and} \qquad \|(u_1, u_2)\| \to \infty \text{ as } \lambda \to 0^+$$

Moreover, $\lambda_\infty = 0$ is the unique bifurcation point from infinity.

Part 3: Non-existence

For $\lambda > \dfrac{\mu_1}{K}$, system $(*)$ has no positive solution.

μ_1 = first Steklov eigenvalue, $\quad K$ = growth constant from $f_i(s) \geq Ks$

Proof Strategy: Theorem 2

1. **Identify the bifurcation point** from the trivial solution via linearization and Jordan diagonalization:
$$\mu_0 = \frac{\mu_1}{\sqrt{f_1'(0)f_2'(0)}}$$
2. **Apply Crandall-Rabinowitz** at $(\mu_0, (0,0))$ to obtain a local branch $\mathscr{C}^+$ of positive solutions bifurcating from the trivial solution.
3. **Non-existence for large λ:** Show that for $\lambda > \mu_1/K$, no positive solution exists. This prevents $\mathscr{C}^+$ from escaping to the right.
4. **Blowup as $\lambda \to 0^+$:** By Rabinowitz global theorem, $\mathscr{C}^+$ must be unbounded, forcing $\|(u_1, u_2)\| \to \infty$ as $\lambda \to 0^+$.

Matrix Formulation

Rewrite system (*) in matrix form:

$$\begin{pmatrix} -\Delta + 1 & 0 \\ 0 & -\Delta + 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega$$

$$\begin{pmatrix} \frac{\partial u_1}{\partial \eta} \\ \frac{\partial u_2}{\partial \eta} \end{pmatrix} = \lambda A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \lambda \begin{pmatrix} R_1(u_2) \\ R_2(u_1) \end{pmatrix} \text{ on } \partial\Omega$$

where $A = \begin{pmatrix} 0 & f_1'(0) \\ f_2'(0) & 0 \end{pmatrix}$.

Key observation: Eigenvalues of A are $\{\sigma, -\sigma\}$ where

$$\sigma := \sqrt{f_1'(0) f_2'(0)}$$

Recall

There exist $\nu_i > 1$ such that

$$f_i(s) = f_i'(0)s + R_i(s) \quad \text{for } s \geq 0$$

where $R_i(s) = O(s^{\nu_i})$ as $s \to 0$

Jordan Canonical Form

We transform the off diagonal matrix to a diagonal matrix

There exists invertible matrix P such that $P^{-1}AP = J$ where:

$$J = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}$$

Explicitly:

$$P = \begin{pmatrix} \frac{1}{1+\zeta} & \frac{1}{1+\zeta} \\ \frac{\zeta}{1+\zeta} & -\frac{\zeta}{1+\zeta} \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} \frac{1+\zeta}{2} & \frac{1+\zeta}{2\zeta} \\ \frac{1+\zeta}{2} & -\frac{1+\zeta}{2\zeta} \end{pmatrix}$$

where $\zeta := \sqrt{\frac{f_2'(0)}{f_1'(0)}}$.

Setting $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} := P^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ transforms system to:

$$\frac{\partial w_1}{\partial \eta} = \lambda\sigma w_1 + \lambda R_1^*(w_1, w_2)$$
$$\frac{\partial w_2}{\partial \eta} = -\lambda\sigma w_2 + \lambda R_2^*(w_1, w_2)$$

When $R_i^* = 0$ (i.e., when the nonlinear remainder terms vanish), the system becomes completely decoupled.

Bifurcation Point Characterization

Proposition

Let $\{(u_{1,n}, u_{2,n})\}$ *be positive solutions with* $\|(u_{1,n}, u_{2,n})\| \to 0$ *and corresponding parameters* $\{\lambda_n\}$. *Then:*

$$\lambda_n \to \mu_0 := \sqrt{\frac{\mu_1}{f_1'(0)f_2'(0)}}$$

Moreover, up to subsequence:

$$\frac{u_{1,n}}{\|(u_{1,n}, u_{2,n})\|} \to \phi_1 \frac{1}{1+\sqrt{f_2'(0)/f_1'(0)}}$$
$$\frac{u_{2,n}}{\|(u_{1,n}, u_{2,n})\|} \to \phi_1 \frac{1}{1+\sqrt{f_1'(0)/f_2'(0)}}$$

in $C^\beta(\overline{\Omega})$.

Proof Sketch of Proposition

Step 1: Set $(v_{1,n}, v_{2,n}) := \frac{(u_{1,n}, u_{2,n})}{\|(u_{1,n}, u_{2,n})\|}$.
Step 2: $(v_{1,n}, v_{2,n})$ satisfies:

$$\int_\Omega \nabla v_{1,n} \nabla \psi_1 + v_{1,n}\psi_1 = \lambda_n \int_{\partial\Omega} \left(f_1'(0) v_{2,n} + \frac{R_1(u_{2,n})}{\|(u_{1,n}, u_{2,n})\|} \right) \psi_1$$
$$\int_\Omega \nabla v_{2,n} \nabla \psi_2 + v_{2,n}\psi_2 = \lambda_n \int_{\partial\Omega} \left(f_2'(0) v_{1,n} + \frac{R_2(u_{1,n})}{\|(u_{1,n}, u_{2,n})\|} \right) \psi_2$$

Step 3: Since $R_i(s) = O(s^{\nu_i})$ with $\nu_i > 1$:

$$\frac{|R_i(u_{j,n})|}{\|(u_{1,n}, u_{2,n})\|} \to 0$$

Step 4: In the limit, obtain linearized system leading to Steklov problem.

Limiting Analysis

After taking limits in the weak formulation:

$$\int_\Omega \nabla v_1 \nabla \psi_1 + v_1 \psi_1 = \lambda \int_{\partial\Omega} f_1'(0) v_2 \psi_1$$
$$\int_\Omega \nabla v_2 \nabla \psi_2 + v_2 \psi_2 = \lambda \int_{\partial\Omega} f_2'(0) v_1 \psi_2$$

In transformed coordinates $(z_1, z_2) = P^{-1}(v_1, v_2)$:

$$-\Delta z_1 + z_1 = 0 \text{ in } \Omega, \quad \frac{\partial z_1}{\partial \eta} = \lambda \sigma z_1 \text{ on } \partial\Omega$$
$$-\Delta z_2 + z_2 = 0 \text{ in } \Omega, \quad \frac{\partial z_2}{\partial \eta} = -\lambda \sigma z_2 \text{ on } \partial\Omega$$

Key insight: Second equation has only trivial solution $z_2 = 0$. The first equation gives $\lambda = \frac{\mu_1}{\sigma} = \sqrt{\frac{\mu_1}{f_1'(0) f_2'(0)}}$ and $z_1 = c\phi_1$.

Crandall-Rabinowitz Application

Theorem

Under hypothesis (H_0), there exists a connected component $\mathscr{C}^+ \subset \Sigma$ of positive solutions emanating from $(\mu_0, (0,0))$ where

$$\mu_0 = \frac{\mu_1}{\sqrt{f_1'(0) f_2'(0)}}$$

Proof outline. Rewrite $(*)$ in matrix form:

$$\begin{pmatrix} -\Delta + 1 & 0 \\ 0 & -\Delta + 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega, \qquad \begin{pmatrix} \frac{\partial u_1}{\partial \eta} \\ \frac{\partial u_2}{\partial \eta} \end{pmatrix} = \lambda A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \lambda \begin{pmatrix} \mathcal{R}_1(u_2) \\ \mathcal{R}_2(u_1) \end{pmatrix} \text{ on } \partial\Omega$$

This allows us to define the operator
$\mathscr{F}(\lambda, (u_1, u_2)) = (u_1, u_2)^t - S\left(\lambda A (u_1, u_2)^t + \lambda (\mathcal{R}_1(u_2), \mathcal{R}_2(u_1))^t\right)$ and check:

1. $\mathscr{F}(\mu_0, (0,0)) = (0,0)$
2. $\dim(\ker(D_{(u_1,u_2)} \mathscr{F}(\mu_0, (0,0)))) = 1$
3. $D_\lambda D_{(u_1,u_2)} \mathscr{F}(\mu_0, (0,0)) \notin \mathrm{Range}(D_{(u_1,u_2)} \mathscr{F}(\mu_0, (0,0)))$

Crandall-Rabinowitz Bifurcation Theorem

Theorem (Crandall-Rabinowitz, 1971)

Let X, Y be Banach spaces and $\mathscr{F} : \mathbb{R} \times X \to Y$ with $\mathscr{F}(\lambda, 0) = 0$. Suppose at $(\lambda_0, 0)$:

1. $\mathscr{F}(\lambda_0, 0) = 0$
2. $\dim(\ker(L_0)) = \mathrm{codim}(\mathrm{Range}(L_0)) = 1, \quad$ where $L_0 := D_u\mathscr{F}(\lambda_0, 0)$
3. **Transversality:** $D_\lambda D_u \mathscr{F}(\lambda_0, 0)[\ker(L_0)] \not\subset \mathrm{Range}(L_0)$

Then $(\lambda_0, 0)$ is a bifurcation point: a curve of nontrivial solutions branches off.

Our operator:

$$\mathscr{F}(\lambda, (u_1, u_2)) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - S\left(\lambda A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \lambda \begin{pmatrix} \mathcal{R}_1(u_2) \\ \mathcal{R}_2(u_1) \end{pmatrix}\right)$$

where $S = \Gamma \circ T$ is the Neumann-to-Dirichlet operator and $A = \begin{pmatrix} 0 & f_1'(0) \\ f_2'(0) & 0 \end{pmatrix}$.

Bifurcation point: $\mu_0 = \dfrac{\mu_1}{\sigma}, \quad \sigma := \sqrt{f_1'(0) f_2'(0)}$

Condition 1

Condition 1: $\mathscr{F}(\mu_0, (0,0)) = (0,0)$ ✓

$(0,0)$ is the trivial solution for all λ. Immediate.

To see this, note that $\mathscr{F}(\lambda, (0,0)) = (0,0)^t - S(\lambda A(0,0)^t + \lambda(0,0)^t) = 0$ for all λ. In particular at $\lambda = \mu_0$.

Recall $A = \begin{pmatrix} 0 & f_1'(0) \\ f_2'(0) & 0 \end{pmatrix}$

The Line of Trivial Solutions

The set $\{(\lambda, (0,0)) : \lambda \in \mathbb{R}\}$ is always a solution. Bifurcation theory asks: at which λ_0 do **nontrivial** solutions branch off from this line?

Condition 2: What is a Simple Eigenvalue?

Simple Eigenvalue

An eigenvalue μ of an operator L is **simple** if:

- $\dim(\ker(L - \mu I)) = 1$ (exactly one independent eigenvector)
- Algebraic multiplicity = Geometric multiplicity = 1

Why does this matter for bifurcation?

- If $\dim(\ker(L_0)) > 1$, multiple directions to branch — no unique bifurcating curve
- If $\dim(\ker(L_0)) = 1$, exactly one direction — guarantees a **clean, unique branch**

For the Steklov Problem

$$-\Delta\varphi + \varphi = 0 \quad \text{in } \Omega, \qquad \frac{\partial\varphi}{\partial\eta} = \mu\varphi \quad \text{on } \partial\Omega$$

The first Steklov eigenvalue μ_1 is **simple**: $\ker = \mathrm{span}\{\varphi_1\}$ with $\varphi_1 > 0$ on $\overline{\Omega}$.

For our system, the challenge is that coupling through A obscures the kernel. We must **diagonalize** to reveal the simple eigenvalue structure.

Condition 2: Diagonalization and Decoupling

Linearization at $(\mu_0, (0,0))$

$$L_0(v_1, v_2) = (v_1, v_2)^t - \mu_0 S(A(v_1, v_2)^t)$$

Diagonalize A: Find P with $P^{-1}AP = J = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}$.

Set $(z_1, z_2)^t = P^{-1}(v_1, v_2)^t$. The system **decouples**:

z_1-**equation** (Steklov):

$$\begin{cases} -\Delta z_1 + z_1 = 0 & \text{in } \Omega \\ \dfrac{\partial z_1}{\partial \eta} = \mu_1 z_1 & \text{on } \partial\Omega \end{cases}$$

μ_1 is **simple** $\Rightarrow z_1 = c\varphi_1$

z_2-**equation** (Robin):

$$\begin{cases} -\Delta z_2 + z_2 = 0 & \text{in } \Omega \\ \dfrac{\partial z_2}{\partial \eta} = -\mu_1 z_2 & \text{on } \partial\Omega \end{cases}$$

Negative coefficient $\Rightarrow z_2 \equiv 0$

Kernel in z-variables

So in the (z_1, z_2)-variables, the kernel vector is $(z_1, z_2) = (c\varphi_1, 0)$.
We now need to convert back to the original (v_1, v_2)-variables via P.

Condition 2: Back to Original Variables

Recall $(z_1, z_2)^t = P^{-1}(v_1, v_2)^t$, so $(v_1, v_2)^t = P(z_1, z_2)^t$. Substituting $(z_1, z_2) = (c\varphi_1, 0)$:

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = P \begin{pmatrix} c\varphi_1 \\ 0 \end{pmatrix} = c\varphi_1 \, P \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The first column of P is $\dfrac{1}{1+\zeta}\binom{1}{\zeta}$, so:

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{c\varphi_1}{1+\zeta} \begin{pmatrix} 1 \\ \zeta \end{pmatrix} =: c \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

where $\zeta = \sqrt{\dfrac{f_2'(0)}{f_1'(0)}}$ and:

$$\phi_1 = \frac{\varphi_1}{1+\zeta}, \qquad \phi_2 = \frac{\zeta\,\varphi_1}{1+\zeta}$$

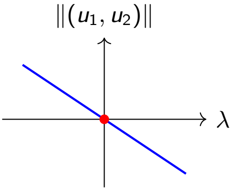
Note: if $f_1'(0) = f_2'(0)$ then $\zeta = 1$ and $\phi_1 = \phi_2 = \varphi_1/2$.

$\ker(L_0) = \mathrm{span}\{(\phi_1, \phi_2)\}$, $\dim(\ker(L_0)) = 1$ ✓ ϕ_1 and ϕ_2 are both multiples of φ_1, weighted by the ratio ζ of the two nonlinearities at zero.

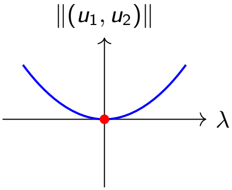
Condition 3: Transversality

What Transversality Means

The parameter λ must move the linearization transversally across the kernel — the bifurcation genuinely crosses rather than touches and bounces back.



Transversal
(crosses through)



Tangent
(touches, bounces back)

What We Need to Show

$$L_1(\phi_1, \phi_2) \notin \mathrm{Range}(L_0), \qquad L_1(v_1, v_2) := -S(A(v_1, v_2)^t)$$

Condition 3: Our Problem Satisfies Transversality

Proof by contradiction. Assume $\gamma L_1(\phi_1, \phi_2) \in \mathrm{Range}(L_0)$ for some $\gamma \neq 0$.

After the Jordan change of variables $(z_1, z_2) = P^{-1}(w_1, w_2)^t$, this forces:

$$\begin{cases} -\Delta z_1 + z_1 = 0 \quad \text{in } \Omega, \quad \dfrac{\partial z_1}{\partial \eta} = \mu_1 z_1 - \gamma\sigma\varphi_1 \quad \text{on } \partial\Omega, \\ -\Delta z_2 + z_2 = 0 \quad \text{in } \Omega, \quad \dfrac{\partial z_2}{\partial \eta} = -\mu_1 z_2 \quad \text{on } \partial\Omega. \end{cases}$$

Test the first equation with φ_1:

$$\mu_1 \int_{\partial\Omega} z_1\varphi_1 = \int_\Omega \nabla z_1 \nabla \varphi_1 + \int_\Omega z_1\varphi_1 = \mu_1 \int_{\partial\Omega} z_1\varphi_1 - \gamma\sigma \int_{\partial\Omega} \varphi_1^2$$

$$\Rightarrow \quad \gamma\sigma \int_{\partial\Omega} \varphi_1^2 = 0$$

Contradiction ×

$\sigma = \sqrt{f_1'(0) f_2'(0)} \neq 0$ and $\displaystyle\int_{\partial\Omega} \varphi_1^2 > 0 \quad \Rightarrow \quad \gamma = 0.$

Conclusion: All three C-R conditions hold $\Rightarrow (\mu_0, (0,0))$ is a bifurcation point. By Rabinowitz's global theorem, $\mathscr{C}^+$ is unbounded in $\mathbb{R} \times (C(\overline{\Omega}))^2$. ✓

Direction of $\mathscr{C}^+$ at μ_0

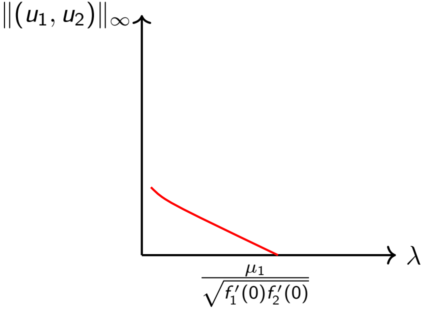


Figure: Bifurcation to the left

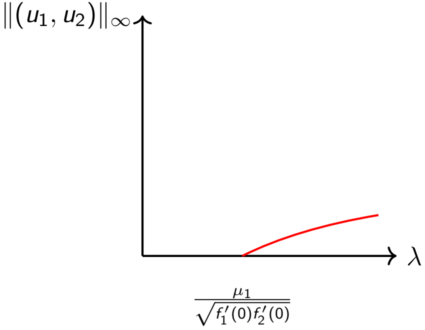


Figure: Bifurcation to the right

Branch can be unbounded in λ direction or $\|(u_1, u_2)\|_\infty$ direction.

Bifurcation Direction Analysis

Define:

$$\nu := \min\{\nu_1, \nu_2\}, \quad \underline{R_i} := \liminf_{s \to 0^+} \frac{R_i(s)}{s^\nu}, \quad \overline{R_i} := \limsup_{s \to 0^+} \frac{R_i(s)}{s^\nu}$$

$$\underline{R_0} := \frac{1}{2}\left[\frac{\zeta}{1+\zeta}\right]^{\nu-1} \underline{R_1} + \frac{1}{2}\left[\frac{1}{1+\zeta}\right]^{\nu-1} \underline{R_2}$$

$$\overline{R_0} := \frac{1}{2}\left[\frac{\zeta}{1+\zeta}\right]^{\nu-1} \overline{R_1} + \frac{1}{2}\left[\frac{1}{1+\zeta}\right]^{\nu-1} \overline{R_2}$$

where $\zeta = \sqrt{\frac{f_2'(0)}{f_1'(0)}}$.

Recall from Hypothesis H_0

There exist $\nu_i > 1$ such that

$$f_i(s) = f_i'(0)s + R_i(s) \quad \text{for } s \geq 0$$

where $R_i(s) = O(s^{\nu_i})$ as $s \to 0$

Bifurcation Direction Theorem

Theorem

Assume nonlinearities satisfy (H_0). *Then:*

1. **(Bifurcation to the left)** *If* $\underline{R_0} > 0$, *then bifurcation occurs to the left:* $\lambda < \mu_0$ *in neighborhood of* μ_0.
2. **(Bifurcation to the right)** *If* $\overline{R_0} < 0$, *then bifurcation occurs to the right:* $\lambda > \mu_0$ *in neighborhood of* μ_0.

Lemma

For solutions $(u_{1,n}, u_{2,n})$ *with* $\lambda_n \to \mu_0 = \dfrac{\mu_1}{\sqrt{f_1'(0) f_2'(0)}}$, $\|(u_{1,n}, u_{2,n})\| \to 0$:

$$\frac{\mu_0}{\sigma}\underline{R_0}\frac{\int_{\partial\Omega}\phi_1^{1+\nu}}{\int_{\partial\Omega}\phi_1^2} \le \liminf_{n\to\infty}\frac{\mu_0-\lambda_n}{\|(u_{1,n}, u_{2,n})\|^{\nu-1}} \le \limsup_{n\to\infty}\frac{\mu_0-\lambda_n}{\|(u_{1,n}, u_{2,n})\|^{\nu-1}} \le \frac{\mu_0}{\sigma}\overline{R_0}\frac{\int_{\partial\Omega}\phi_1^{1+\nu}}{\int_{\partial\Omega}\phi_1^2}$$

- $\underline{R_0} < 0 \implies \lambda_n < \mu_0$, therefore bifurcation to the left
- $\overline{R_0} > 0 \implies \lambda_n > \mu_0$, therefore bifurcation to the right

Non-existence Result

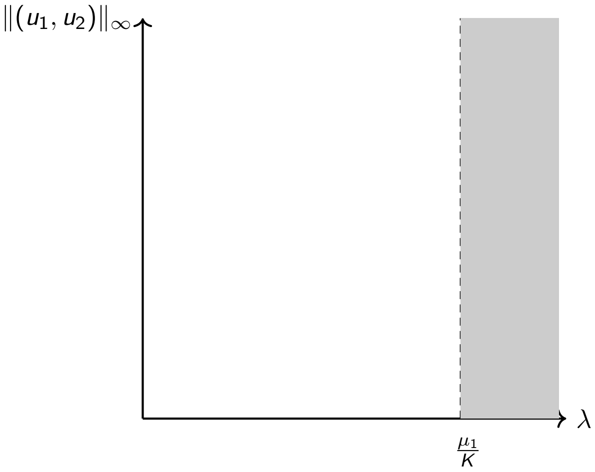


Figure: Non-existence for large λ

Proof of Non-existence for Large λ

Claim: No positive solutions exist for $\lambda > \frac{\mu_1}{K}$.
Proof: Suppose (u_1, u_2) is positive solution for $\lambda > \frac{\mu_1}{K}$.
Test first equation with $\phi_1 > 0$:

$$\begin{aligned} 0 &= \lambda \int_{\partial\Omega} f_1(u_2)\phi_1 - \int_\Omega [\nabla u_1 \nabla \phi_1 + u_1\phi_1] \\ &\geq \lambda K \int_{\partial\Omega} u_2\phi_1 - \mu_1 \int_{\partial\Omega} u_1\phi_1 \quad (\text{using } f_1(s) \geq Ks) \\ &> \mu_1 \int_{\partial\Omega} (u_2 - u_1)\phi_1 \end{aligned}$$

Similarly for second equation:

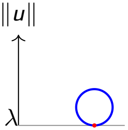
$$0 > \mu_1 \int_{\partial\Omega} (u_1 - u_2)\phi_1$$

This gives $\int_{\partial\Omega}(u_2 - u_1)\phi_1 > 0$ and $\int_{\partial\Omega}(u_1 - u_2)\phi_1 > 0$, contradiction.

Rabinowitz: Why is $\mathcal{C}^+$ Unbounded?

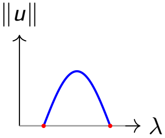
Rabinowitz: The branch $\mathcal{C}^+$ either is unbounded OR meets another bifurcation point.

× **Loop back to same point**



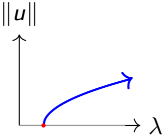
Crandall-Rabinowitz: only **one branch** leaves the bifurcation point — can't re-enter!

× **Join another bifurcation point**



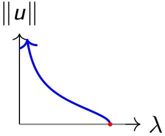
$\mathcal{C}^+$ has **positive solutions**; other bifurcation points have **sign-changing** eigenfunctions.

× **Unbounded in λ**



Nonexistence for large λ rules this out.

✓ **Unbounded in $\|u\|$**

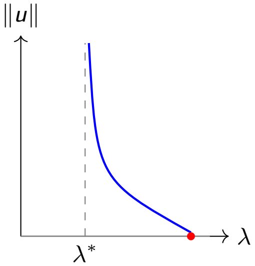


The only remaining possibility!

Where Does $\|u\| \to \infty$ Happen?

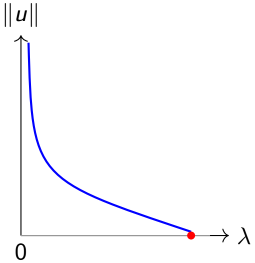
We know $\mathcal{C}^+$ is unbounded in $\|u\|$. But **where**?

× **Blowup at some** $\lambda^* > 0$



A priori bounds: For any $[a_0, b_0]$ with $a_0 > 0$, solutions are uniformly bounded. Blowup **cannot** happen at $\lambda^* > 0$.

✓ **Blowup as** $\lambda \to 0^+$



The **only** possibility! This is **bifurcation from infinity** at $\lambda = 0$.

Conclusion: $\mathcal{C}^+$ connects bifurcation from the trivial solution to bifurcation from infinity at $\lambda = 0$.

Global Bifurcation Diagrams

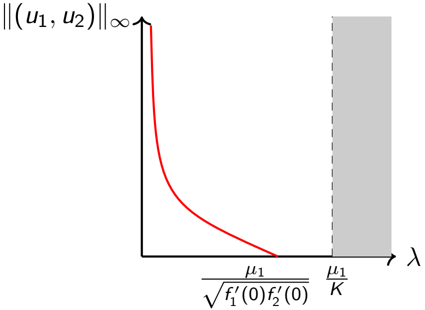


Figure: Bifurcation to the left

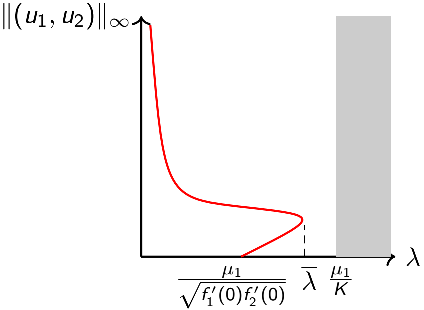


Figure: Bifurcation to the right

Bifurcation to the right enables multiplicity.

Important References

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Future Directions

Current Work in Progress: **PROOF, SL Math in Summer 2026**

Single Equation with Drift Term and Nonlinearity inside the Domain

$$
\begin{aligned}
-\Delta u + c(x)u &= \lambda f(u) \quad \text{in } \Omega \\
\frac{\partial u}{\partial \eta} &= \lambda g(u) \quad \text{on } \partial\Omega
\end{aligned}
$$

Planned Extensions:

Double Phase Problems

$$
\begin{aligned}
-\Delta_p u - \Delta_q u + c_1(x)|u|^{p-2}u + c_2(x)|u|^{q-2}u &= f(x,u) \quad \text{in } \Omega \\
(|\nabla u|^{p-2} + |\nabla u|^{q-2})\frac{\partial u}{\partial \nu} &= g(x,u) \quad \text{on } \partial\Omega
\end{aligned}
$$

Difficulties: Non-uniform ellipticity, variable growth conditions, lack of homogeneity.

Key Questions:

- Regularity theory for variable exponent problems
- Bifurcation analysis when principal part changes type
- Extension to systems with mixed p-q structure

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Questions & Discussion

Thank you for your attention!

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