Differential Equations I (Math 330)

Course Notes: Introduction

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Introduction to Differential Equations

A differential equation is a mathematical equation that relates a function with its derivatives. These equations are essential for modeling dynamic systems and processes in physics, engineering, economics, and many other fields. This course focuses on methods for solving various types of differential equations and understanding their applications.

Definition

Differential Equation: A differential equation is any equation involving an unknown function and one or more of its derivatives.
Ordinary Differential Equation (ODE): If the unknown function in a differential equation is a function of only one variable, the equation is called an ordinary differential equation, or ODE.
Partial Differential Equation (PDE): If the unknown function in a differential equation depends on more than one independent variable, the equation is called a partial differential equation, or PDE.

Classification of Differential Equations

The study of differential equations typically begins with classifying them according to their properties. Complete the following table to practice identifying different types of differential equations:

Application Area	**Equation**	**ODE or PDE**
Population dynamics, economics, logistic growth	$\frac{dP}{dt} = k(P - P^2)$	
Electrical circuits, oscillations, vibrations	$5\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 9x = 2\cos 3t$	
Electrostatics, heat flow, fluid dynamics	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	
Beam deflection in structural engineering	$8\frac{d^4y}{dx^4} = x(1 - x)$	
Nuclear physics, diffusion processes	$\frac{\partial N}{\partial t} = \frac{\partial^2 N}{\partial r^2} + \frac{1}{r}\frac{\partial N}{\partial r} + kN$	

Note

In this course, we will focus exclusively on ordinary differential equations (ODEs).

Key Terminology

Understanding the vocabulary associated with differential equations is essential for discussing and solving them effectively.

Definition

Dependent Variable: The unknown function that we're solving for in a differential equation.
Independent Variable: The variable with respect to which the function is defined.
Parameter: A quantity that remains constant for a specific problem but may vary between different problems.
Order: The highest derivative of the unknown function that appears in the equation.

Practice with Terminology

For each equation below, identify the order, dependent variable, independent variable, and parameters:

Equation	Order	Dependent	Independent	Parameters
$\frac{dP}{dt} = k(P - P^2)$				
$5\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 9x = 2\cos 3t$				
$8\frac{d^4y}{dx^4} = x(1-x)$				
$mx'' + bx' + kx = 2t^5$				
$x''' + 2x'' + x' + 3x = \sin(\omega t)$				

Solutions to Differential Equations

Definition

Solution: A solution of a differential equation is a sufficiently differentiable function that, when substituted into the equation along with its derivatives, satisfies the equation for all values of the independent variable within some interval.

Example

Determine whether each function is a solution to the given differential equation:

1. $y'' - y = 0$, with $y(t) = e^t$
2. $y'' - y = 0$, with $y(t) = \cosh t$
3. $y' - y = 2$, with $y(t) = 3e^t - 2$
4. $2t^2y'' + 3ty' - y = 0$, with $y(t) = \sqrt{t}$ for $t > 0$
5. $y'' + y = \sec t$, with $y(t) = (\cos t)\ln(\cos t) + \sin t$ for $0 < t < \pi/2$

Exponential Growth Models

One of the most fundamental differential equations models exponential growth or decay. This model appears in many applications, from population dynamics to radioactive decay.

The Exponential Growth Equation

From Calculus I (Section 3.8 of Stewart's Calculus), we recall:

> "In general, if $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size $y(t)$ at any time, then
>
> $$\frac{dy}{dt} = k \cdot y(t) \tag{1}$$
>
> where k is a constant."

This model is often called "the law of uninhibited growth" or the "Malthusian model." The general solution to this differential equation is:

$$y(t) = y(0) \cdot e^{kt} \tag{2}$$

Exploring Solutions

Consider the simple differential equation $y' = y$.

A solution to this ODE is $y(t) = e^t$. But is it the only solution?

What about $y(t) = 2e^t$? Or $y(t) = 3e^t$? Or more generally, $y(t) = Ce^t$ for any constant C?

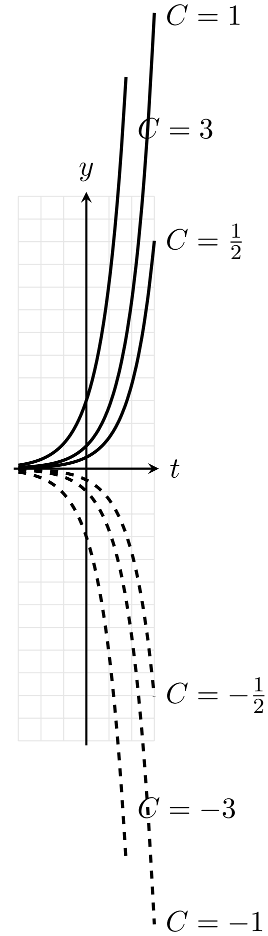

Figure 1: Graph of $y(t) = Ce^t$ for selected values of C

Would the family of curves $y(t) = Ce^t$ "fill up" the entire Cartesian plane if we graphed all possible values of C? What patterns do you notice in this family of solutions?

Types of Solutions

When working with differential equations, we distinguish between different types of solutions based on how completely they characterize the solution space and what additional conditions they satisfy.

Definition

General Solution: A family of solutions containing all possible solutions to a differential equation. These solutions should be non-intersecting and collectively fill the solution space.
Initial Value Problem (IVP): A differential equation with additional conditions specified at a single point. These additional constraints are called initial conditions.
Boundary Value Problem (BVP): A differential equation with additional conditions specified at multiple points. These constraints are called boundary conditions. (Note: BVPs are typically harder to solve than IVPs.)
Particular Solution: A specific solution that satisfies both the differential equation and any additional conditions (initial or boundary).

Example

Determine each of the following if initial value problems or boundary value problems:

1. $x' = x$, $x(0) = 2$
2. $\dfrac{dx}{dt} = x$, $x(0) = -3$
3. $x'' + x = 0$, $x(0) = 1$, $x'(0) = 1$
4. $\dfrac{d^2x}{dt^2} + x = 0$, $x(0) = 0$, $x\left(\dfrac{\pi}{2}\right) = 2$
5. $x' = 2t - 5$, $x(1) = 1$
6. $x' = 2x - 5$, $x(1) = 1$

Homework Assignment

1. For each of the following differential equations, determine whether it is an ordinary differential equation (ODE) or a partial differential equation (PDE):

   (i) $\dfrac{d^2y}{dt^2} - 2y\dfrac{dy}{dt} + 2y = 0$

   (ii) $\dfrac{dy}{dt} = \dfrac{y(2-3t)}{(1-3y)}$

   (iii) $\dfrac{dx}{dt} = k(4-x)(1-x)$, where k is a constant

   (iv) $\dfrac{\partial u}{\partial t} - D\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) = 0$

   (v) $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 3\cos(\omega t)$

   (vi) $\dfrac{\partial^2\psi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\psi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\psi}{\partial\theta^2} = 0$

   (vii) $y''' + 4yy'' - 3y' + 2y = \sin(t)$

   (viii) $\dfrac{d^3z}{dx^3} + x^2\dfrac{dz}{dx} = e^x$

2. For each of the ODEs identified in problem 1, determine the order, dependent variable, independent variable, and any parameters.

3. Determine whether the given function is a solution to the given differential equation:

   (i) $x'' + x = 0$, with possible solution $x(t) = 2\cos t - 3\sin t$

   (ii) $\dfrac{d^2y}{dx^2} + y = x^2 + x + 1$, with possible solution $y(x) = \sin x + x^2 + x + 1$

   (iii) $\dfrac{d^2\theta}{dt^2} - \theta\dfrac{d\theta}{dt} + 3\theta = -2e^{2t}$, with possible solution $\theta(t) = 2e^{3t} - e^{2t}$

   (iv) $y'' + 4y = -5e^{-x}$, with possible solution $y(x) = 3\sin 2x - e^{-x}$

   (v) $y'' - 4y' + 4y = 0$, with possible solution $y(x) = (c_1 + c_2x)e^{2x}$ where $c_1 = 3$ and $c_2 = -2$

   (vi) $x'' - 6x' + 9x = 0$, with possible solution $x(t) = t^2e^{3t}$

4. Determine whether the given relation is an implicit solution to the given differential equation. Assume that the relationship defines y as a function of x and use implicit differentiation:

(i) $\dfrac{dy}{dx} = \dfrac{2y-1}{x}$, with possible solution $y - \ln y = x^2 + 1$

(ii) $\dfrac{dy}{dx} = e^{-xy} - ye^{-xy} + x$, with possible solution $e^{xy} + y = x - 1$

(iii) $\dfrac{dy}{dx} = \dfrac{xy - y^2}{x^2}$, with possible solution $\dfrac{y}{x} + \ln|x| = C$ where $C = 2$

(iv) $\dfrac{dy}{dx} = \dfrac{y^2}{x^3}$, with possible solution $\dfrac{1}{y} = \dfrac{1}{2x^2} + C$ where $C = -\dfrac{1}{8}$

5. Classify each of the following as an initial value problem (IVP) or a boundary value problem (BVP):

   (i) $y'' + y = 0$, $y(0) = 0$, $y'(0) = 1$

   (ii) $y'' + y = 0$, $y(0) = 0$, $y(\pi) = 0$

   (iii) $x'' + 9x = \cos(3t)$, $x(0) = 1$, $x'(0) = 0$

   (iv) $\dfrac{d^2y}{dx^2} + 4y = x$, $y(0) = 0$, $y(2) = 3$

   (v) $y^{(4)} - 16y = 0$, $y(0) = y'(0) = 0$, $y(1) = 1$, $y'(1) = 0$

6. Prove that if $y_1(t)$ and $y_2(t)$ are both solutions to the linear homogeneous equation $y'' + p(t)y' + q(t)y = 0$, then $C_1y_1(t) + C_2y_2(t)$ is also a solution for any constants C_1 and C_2.

Differential Equations I (Math 330)

Course Notes: First-Order Separable DE

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First-order Differential Equations

Technically, from our definition of a differential equation, we can deduce that a first-order differential equation is any equation involving an unknown function and its first derivative. In its most general sense, this concept can be written as $F(t, x, x') = 0$, for some unknown function F. For the solving techniques presented in this chapter, we assume that the equation can be solved explicitly for x', and thus, we consider differential equations that can be put in the form:

$$x' = f(t, x),$$

where f denotes an arbitrary function in two variables.

Separable First-order Differential Equations

A differential equation is separable when it can be written in the form

$$\frac{dx}{dt} = g(t)h(x);$$

that is, when the function $f(t, x)$ can be factored into a product of a function of t times a function of x.

Examples

Determine which of the following first-order differential equations are separable.

1. $x' = 3x + 1$
2. $x' = 2t - 7$
3. $x' = 3x - 2t$
4. $x' = tx + 2t$
5. $x' = \cos(x + t)$
6. $x' = e^{x+t}$
7. $x' = xt^2 + t^2 - x - t$

Method to solve a separable first-order differential equation:

Start of problem: $x' = f(t, x)$

1. Simplify the equation so that it can be written in the form $\frac{dx}{dt} = g(t)h(x)$.
2. Multiply both sides by dt and divide both sides by $h(x)$, and insert appropriate integral signs to obtain

$$\int \frac{1}{h(x)} dx = \int g(t) dt.$$

3. Integrate both sides of the equation; that is, find any function $H(x)$ such that $H'(x) = \frac{1}{h(x)}$ and any function $G(t)$ such that $G'(t) = g(t)$. Include a constant of integration on the side of the independent variable so that the equation can be written as

$$H(x) = G(t) + C.$$

4. If possible, solve this equation explicitly for x so that it can be written as $x(t) = \ldots$

Examples

Solve the following separable equations and initial value problems.

1. $x' = \frac{t}{x}$
2. $x' = tx + 5$
3. $x' = x \cos t$
4. $x' = (t+1)(\cos x)^2$
5. $x' = \frac{t}{xe^{t+2x}}$
6. $x' = tx + 1, x(0) = 2$

Homework Assignment

Solve the following differential equations and initial value problems using the method of separation of variables, if possible. If an equation is not separable, explain why this method cannot be used.

1. $x' = \frac{x}{t}$
2. $x' = -x^2 \sin t$
3. $x' = 4x^2 - 3t + 1$
4. $x' = \frac{t+tx^2}{e^{t^2x}}$
5. $x' = \frac{3t^2-1}{3+2x}$
6. $x' = tx + 7$
7. $x' = (2+t)(3+x)$
8. $x' = t, x(0) = 3$
9. $x' = (1+t)(2+x), x(0) = -1$
10. $x' = (1+x^2)\tan t, x(0) = \sqrt{3}$
11. $x' = x \cos t, x(0) = 1$
12. $x' = 2x(1-x), x(0) = 1/2$

Differential Equations I (Math 330)

Course Notes: First-Order Linear DE

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Linear First-order Differential Equations

We proceed beyond the intuitive calculus method of separable variables by considering the special case when the differential equation is in the form

$$a_1(t)x'(t) + a_0(t)x(t) = b(t), \tag{1}$$

where the functions a_0, a_1, and b are arbitrary functions of t. Since we are assuming a first-order equation, we know that $a_1(t) \not\equiv 0$ and this allows us to divide the expression by a_1 and give us the standard form of the equation we will need to apply our technique.

Definitions

Linear Differential Equation in Standard Form: A first-order linear differential equation in standard form is an equation that can be written as

$$x'(t) + p(t)x(t) = q(t), \tag{2}$$

for some functions $p(t)$ and $q(t)$. If $q(t) \equiv 0$, it is called a homogeneous linear equation.

Examples

Determine which of the following first-order differential equations are linear. If so, are they also homogeneous?

1. $x' = 3x + 1$
2. $x' = 2t - 7$
3. $x' = 3x - 2t$
4. $x' = tx + 2t$
5. $t^2x' = \sin t + x$
6. $(t^2 - 1)x' = tx - x$
7. $xx' + t^2x = \sin t$
8. $t^2x' + 3tx = t^4 \ln t + 1$

Example (Method Illustration, simple case)

Consider the first-order linear differential equation given as follows

$$x' + x = 1 \tag{3}$$

where $p(x) \equiv 1$ and $q(x) \equiv 1$. We seek a way to turn this equation into something that is nearly separable; that is, we seek a way to rewrite the left-hand-side of the equation so that it is an application of the product rule for derivatives. We accomplish this by multiplying the entire equation by a function, call it $\alpha(t)$.

$$\begin{aligned} \alpha(t)(x' + x) &= \alpha(t) \cdot 1 \\ \alpha(t)x' + \alpha(t)x &= \alpha(t) \\ (f(t) \cdot x)' &= \alpha(t) \end{aligned} \tag{4}$$

Also note that

$$\begin{aligned}(f(t)\cdot x)' &= \alpha(t)\\ f'(t)x + f(t)x' &= \alpha(t)\\ f(t)x' + f'(t)x &= \alpha(t)\end{aligned} \tag{5}$$

Notice that we need a function where $f(t) = \alpha(t)$ and $f'(t) = \alpha(t)$. We know this function to be $\alpha(t) = e^t$. We use this as the integrating factor for the linear equation given above to yield

$$\begin{aligned}e^t(x' + x = 1)\\ e^t x' + e^t x = e^t\\ (e^t x)' = e^t\end{aligned} \tag{6}$$

We then integrate this equation, similar to how we solved separable equations,

$$\begin{aligned}\int (e^t x)' &= \int e^t dt\\ e^t x &= e^t + c\\ x &= 1 + ce^{-t}\end{aligned} \tag{7}$$

Example (Method Illustration, general case)

Now, consider the first-order linear differential equation given as follows

$$x' + p(t)x = q(t). \tag{8}$$

We proceed through a similar sequence as in the simple case. Call our integrating factor $\mu(t)$ this time.

$$\begin{aligned}\mu(t)(x' + p(t)x) &= \mu(t)q(t)\\ \mu(t)x' + \mu(t)p(t)x &= \mu(t)q(t)\end{aligned} \tag{9}$$

And by similar reasoning, we need a function that satisfies $f(t) = \mu(t)$ and $f'(t) = \mu(t)p(t)$. We can deduce that it should be exponential of some type so that when differentiating, the chain rule yields the derivative of the exponent to be $p(t)$, i.e.

$$f(t) = \mu(t) = e^{\int p(t)dt} \tag{10}$$

so that

$$f'(t) = e^{\int p(t)dt} \cdot \left(\int p(t)dt\right)' = e^{\int p(t)dt} \cdot p(t) = \mu(t)p(t). \tag{11}$$

Thus, we may proceed with our solution

$$\begin{aligned}\mu(t)x' + \mu(t)p(t)x &= \mu(t)q(t)\\ (\mu(t)x)' &= \mu(t)q(t)\\ \mu(t)x &= \int [\mu(t)q(t)]dt\end{aligned} \tag{12}$$

And, after integrating the right-hand-side, we can divide by $\mu(t)$ to obtain our general solution solved explicitly for $x(t)$.

Definitions

Integrating Factor: A function $\mu(t)$ such that multiplying the equation by μ makes it integrable is called an integrating factor for the differential equation. In general, an integrating factor for a first-order linear differential equation in standard form is given by

$$\mu(t) = e^{\int p(t)dt} \tag{13}$$

Method to solve a first-order linear differential equation:

Start of problem:

$$a_1(t)x'(t) + a_0(t)x(t) = b(t) \tag{14}$$

1. Simplify the equation so that it can be written in standard form.
2. Multiply both sides by the integrating factor $\mu(t)$.
3. Simplify the equation via the product rule on the left-hand-side of the equation and any allowable algebraic simplification on the right-hand-side of the equation.
4. Integrate both sides of the resulting equation with respect to t, making sure to add the constant of integration on the right-hand-side.
5. Divide both sides by $\mu(t)$ to obtain the explicit solution $x(t)$. Note that the right-hand-side might require simplification at this point.

Examples

Solve the following linear equations and initial value problems.

1. $x' - x - e^{3t} = 0$
2. $x' = \frac{x}{t} + 2t + 1$
3. $\frac{dx}{dt} + \frac{3x}{t} + 2 = 3t, x(1) = 1$

Homework Assignment

For each differential equation below, determine whether it is linear or nonlinear. If linear, state whether it is homogeneous or nonhomogeneous.

1. $x' + 3x\sin(t) = t^2$
2. $tx' + e^t x = \ln(t)$
3. $(t^2+1)x' - 5tx = \cos(t)$
4. $x' = x^2 + t$
5. $x' + \frac{x}{t-1} = t^3$
6. $x' = \sqrt{x} + t^2$

Solve the following differential equations and initial value problems using the method involving integrating factors for linear equations.

1. $x' = 2x + 1$

2. $x' + 2x = e^{-2t} \sin t$
3. $tx' + x = 3t^2 - t$
4. $x' + 2tx = 3t$
5. $x' = -x + e^{2t}, x(0) = 1$
6. $x' + 2x = e^{-2t} \cos(t), x(0) = -1$
7. $x' + 2tx = 3t, x(0) = 4$
8. $x' = t^2 e^{-4t} - 4x$
9. $x' + x \tan t = \sec t$
10. $x' + 4x - e^{-t} = 0, x(0) = \frac{4}{3}$
11. $(\sin t)\frac{dx}{dt} + x(\cos t) = t \sin t, x\left(\frac{\pi}{2}\right) = 2$

Consider the following attempt to connect separable equations to linear equations.

1. We now know what it means for a differential equation to be separable and what it means for a differential equation to be linear. Both methods of solving these types of equations share similar techniques. This then begs the questions:
   (a) Under what conditions is a separable equation solvable via the linear equation technique?
   (b) Under what conditions is a linear equation solvable via the separable equation technique?
   (c) Is there a general class of problem that is solvable using either technique?

Differential Equations I (Math 330)

Course Notes: First-Order Exact Equations

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Exact Equations

As an extension to the linear equations method and specifically the integrating factor concept, we consider a special case when the differential equation is in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0. \tag{1}$$

Most first-order equations can be put into this form via simple algebraic manipulation, however only a select few of these can be called exact.

Definitions

[Exact Equations] A first-order differential equation $y' = f(x,y)$ is called an exact equation if it can be written in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{2}$$

where the functions $M(x,y) = \frac{\partial F}{\partial x}$ and $N(x,y) = \frac{\partial F}{\partial y}$ for some differentiable function $F(x,y)$.

In (multivariate) calculus, it is shown that if F is a twice continuously differentiable function of two variables, then the second-order mixed partial derivatives $\frac{\partial^2 F}{\partial y \partial x}$ and $\frac{\partial^2 F}{\partial x \partial y}$ are equal. If $M(x,y) = \frac{\partial F}{\partial x}$ and $N(x,y) = \frac{\partial F}{\partial y}$, then

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) \equiv \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial N}{\partial x}. \tag{3}$$

We use this as our "check" for exactness given an equation of the appropriate form.

Example (Exactness Check)

Check whether the following differential equation is exact.

$$(x^2 + y)\,dx + (x - \sin y)\,dy = 0$$

Begin by defining M and N:

$$\begin{aligned} \text{Let } M(x,y) &= x^2 + y \\ \text{Let } N(x,y) &= x - \sin y \end{aligned}$$

Then, perform the appropriate differentiations.

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x^2 + y) = 1 \\ \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x - \sin y) = 1 \end{aligned}$$

Therefore, this equation is exact.

Examples

Determine which of the following first-order differential equations are exact.

1. $(3x^2 - 2x + 2)\,dx + (6y^2 - x^2 + 3)\,dy = 0$
2. $(e^x \sin y - 2y \sin x)\,dx + (e^x \cos y + 2\cos x)\,dy = 0$
3. $(x \ln y + \frac{x}{y})\,dx + (y \ln x + \frac{x}{y})\,dy = 0,\ x > 0,\ y > 0$
4. $\frac{dy}{dx} = -\frac{ax + by + c}{dx + ey + f}$

Method to solve an exact equation:

Start of problem:

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

1. Check for exactness, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
2. Set $\frac{\partial F}{\partial x} = M(x, y)$ and integrate once with respect to x to get

$$F(x, y) = \int M(x, y)\, dx + Q(y). \tag{4}$$

   Note that $Q(y)$ represents the "constant" of integration with respect to x, and the integration is done as though x is the only variable, and y is a parameter.

3. Differentiate the function F found in step (2), partially with respect to y, and set the result equal to $N(x, y)$.
4. The result of steps (2) and (3) yields an equation that defines $Q'(y)$ as a function of y only. Antidifferentiate $Q'(y)$ to obtain $Q(y)$.
5. The function from step (2) with the value of $Q(y)$ from step (4) will provide an implicit solution of the given exact differential equation,

$$F(x, y) = C. \tag{5}$$

Note that this method also works if you begin by setting $\frac{\partial F}{\partial y} = N(x, y)$ and proceeding in this manner.

Example (Exactness Method)

Solve the following differential equation using this exact method.

$$(x^2 + y)\, dx + (x - \sin y)\, dy = 0 \tag{6}$$

We know this equation is exact from 'Example (Exactness Check)' so we proceed with the method. Recall that we made the following definitions:

$$\text{Let } M(x, y) = x^2 + y$$
$$\text{Let } N(x, y) = x - \sin y$$

Option 1 (begin with M):

$$\begin{aligned} F(x, y) &= \int (x^2 + y)\, dx + Q(y) \\ &= \frac{1}{3}x^3 + xy + Q(y) \end{aligned}$$

Now, differentiate F with respect to y,

$$\begin{aligned} \frac{\partial F}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{1}{3}x^3 + xy + Q(y)\right) \\ &= x + Q'(y) \equiv N(x, y) \end{aligned}$$

Thus,

$$\begin{aligned} x + Q'(y) &= x - \sin y \\ \Rightarrow Q'(y) &= -\sin y \end{aligned}$$

And integrate with respect to y to obtain

$$Q(y) = \cos y$$

Therefore, the implicit solution becomes

$$\frac{1}{3}x^3 + xy + \cos y = C$$

Option 2 (begin with N):

$$\begin{aligned} F(x,y) &= \int (x - \sin y)\, dy + Q(x) \\ &= xy - \cos y + Q(x) \end{aligned}$$

Now, differentiate F with respect to x,

$$\begin{aligned} \frac{\partial F}{\partial x} &= \frac{\partial}{\partial x}(xy - \cos y + Q(x)) \\ &= y + Q'(x) \end{aligned}$$

Thus,

$$\begin{aligned} y + Q'(x) &= x^2 + y \\ \Rightarrow Q'(x) &= x^2 \end{aligned}$$

And integrate with respect to x to obtain

$$Q(x) = \frac{1}{3}x^3$$

Therefore, the implicit solution becomes

$$xy - \cos y + \frac{1}{3}x^3 = C$$

Examples

Solve the following differential equations using the exact method.

1. $(2x + y)\, dx + (x - 2y)\, dy = 0$
2. $(\cos x)\, dx - (\sin x - e^y)\, dy = 0$
3. $(1 + 2y^2 x)\, dx + (2yx^2 - \cos y)\, dy = 0,\ y(1) = \pi$

Homework Assignment

Solve the following differential equations using the exact method, if possible. If the equation cannot be solved using this method, explain why not.

1. $(2x + 3)\, dx + (2y - 2)\, dy = 0$
2. $(2xy^2 + 2y)\, dx + (2x^2 y + 2x)\, dy = 0$
3. $(e^x \sin y + 3y)\, dx - (3x - e^x \sin y)\, dy = 0$
4. $(1 + \ln y)\, dx + \left(\frac{x}{y}\right)\, dy = 0$
5. $(\cos x)\, dx - (y \sin x - e^y)\, dy = 0$
6. $(ye^{xy} - 1)\, dx + (xe^{xy} + 2)\, dy = 0,\ y(1) = 1$
7. $(e^{x+y} + xe^{x+y})\, dx + (xe^{x+y} + 2)\, dy = 0,\ y(0) = -1$
8. $\frac{dy}{dx} = -\frac{ax+by+c}{dx+ey+f}$, where a, b, c, d, e, f are constants

Differential Equations I (Math 330)

Course Notes: First-Order Bernouilli Equations

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Bernoulli Equations

Introduction

Bernoulli equations represent an important class of nonlinear differential equations that can be transformed into linear equations through a clever substitution. They are named after the Swiss mathematician Jacob Bernoulli and appear frequently in various applications of mathematics, physics, and engineering.

Definition and Key Concepts

[Bernoulli Equations] A first-order differential equation is classified as a Bernoulli equation if it can be expressed in the form:

$$x' + p(t)x = q(t)x^n, \quad n \neq 0, 1 \tag{1}$$

where $p(t)$ and $q(t)$ are continuous functions of the independent variable t, and n is a real number.

It's worth noting that when $n = 0$ or $n = 1$, the equation reduces to a linear differential equation. For $n = 0$, we get $x' + p(t)x = q(t)$, which is clearly linear. Similarly, for $n = 1$, we have $x' + p(t)x = q(t)x \implies x' + (p(t) - q(t))x = 0$, which is also linear, in particular separable

Solution Strategy

For values of n other than 0 and 1, we employ a substitution technique to transform the Bernoulli equation into a linear one. The key substitution is:

$$v = x^{1-n} \tag{2}$$

Taking the derivative with respect to t:

$$\begin{aligned} v' &= \frac{d}{dt}(x^{1-n}) \\ &= (1-n)x^{-n}x' \end{aligned}$$

Rearranging to isolate x':

$$x' = \frac{v'}{(1-n)x^{-n}}$$

Now, let's substitute this into our original Bernoulli equation:

$$\frac{v'}{(1-n)x^{-n}} = -p(t)x + q(t)x^n$$

Multiplying both sides by x^{-n}:

$$\begin{aligned} \frac{v'}{(1-n)} &= -p(t)x^{1-n} + q(t) \\ \frac{v'}{(1-n)} &= -p(t)v + q(t) \end{aligned}$$

Multiplying both sides by $(1-n)$:

$$v' + (1-n)p(t)v = (1-n)q(t)$$

This is now a linear first-order differential equation in terms of v and t. We can solve it using standard methods for linear equations, then convert back to find x using the relation $x = v^{\frac{1}{1-n}}$.

Step-by-Step Solution Process

To solve a Bernoulli equation:

1. Verify that the given equation is indeed a Bernoulli equation and identify the value of n.
2. Make the substitution $v = x^{1-n}$.
3. Derive the corresponding linear differential equation in terms of v.
4. Solve this linear equation using appropriate techniques (e.g., integrating factor method).
5. Convert the solution back to the original variable using $x = v^{\frac{1}{1-n}}$.

Example: Solving a Bernoulli Equation

Let's solve the following Bernoulli equation:

$$\frac{dx}{dt} + \frac{x}{t} = t^2 x^2$$

First, we rearrange it into the standard form:

$$x' = -\frac{x}{t} + t^2 x^2$$

Comparing with $x' + p(t)x = q(t)x^n$, we identify:

$$\begin{aligned} p(t) &= \frac{1}{t} \\ q(t) &= t^2 \\ n &= 2 \end{aligned}$$

Since $n = 2$, we make the substitution:

$$\begin{aligned} v &= x^{1-2} = x^{-1} \\ v' &= (1-2)x^{-2}x' = -x^{-2}x' \end{aligned}$$

This gives us:

$$\begin{aligned} -v' &= -x^{-2}\left(-\frac{x}{t} + t^2x^2\right) \\ -v' &= \frac{x^{-1}}{t} - t^2 \\ -v' &= \frac{v}{t} - t^2 \\ v' &= -\frac{v}{t} + t^2 \end{aligned}$$

Now we have a linear equation in v:

$$v' + \frac{v}{t} = t^2$$

We solve this using the integrating factor method. The integrating factor is:

$$\mu = e^{\int \frac{1}{t}dt} = e^{\ln t} = t$$

Multiplying the differential equation by $\mu = t$:

$$\begin{aligned} tv' + v &= t^3 \\ \frac{d}{dt}(tv) &= t^3 \end{aligned}$$

Integrating both sides:

$$tv = \int t^3 dt = \frac{t^4}{4} + C$$
$$v = \frac{t^3}{4} + \frac{C}{t}$$

Converting back to our original variable:

$$x^{-1} = \frac{t^3}{4} + \frac{C}{t}$$
$$x = \frac{1}{\frac{t^3}{4} + \frac{C}{t}} = \frac{t}{\frac{t^4}{4} + C}$$

Therefore, the general solution to the original Bernoulli equation is:

$$x(t) = \frac{t}{\frac{t^4}{4} + C}$$

Interesting Observation

It's noteworthy that the constant solution $x(t) = 0$ also satisfies the original Bernoulli equation. However, this solution is not captured by our substitution method because $v = x^{-1}$ is undefined when $x = 0$. This highlights a limitation of the substitution technique—it may miss certain singular solutions.

Practice Examples

Try solving these Bernoulli equations:

1. $\frac{dx}{dt} - x = e^{2t}x^3$
2. $\frac{dx}{dt} = 2x - t^2x^2$
3. $\frac{dx}{dt} + \frac{x}{t-2} = 5(t-2)x^{1/2}$
4. $\frac{dx}{dt} + tx^3 + \frac{3x}{2t} = 0$

Homework Assignment

Part A: Bernoulli Equations

For each of the following Bernoulli equations, identify the values of $p(t)$, $q(t)$, and n. Apply the substitution $v = x^{1-n}$ to transform the equation to a linear form. Solve the resulting linear equation and then find the general solution of the original equation. Include any special cases or singular solutions that may exist.

1. $x' = 2x\left(1 - \frac{x}{4}\right)$
2. $x' + \frac{x}{t} = e^t x^{-2}$
3. $t^2x' + 2tx - x^3 = 0$, $t > 0$
4. $\frac{dx}{dt} = \frac{-2t+2x}{2+t^2}$
5. $\frac{dx}{dt} + \frac{x}{3t} + x^3 t = 0$
6. $\frac{dx}{dt} + 2x = 3x^2 \sin t$

7. $\frac{dx}{dt} - \frac{t}{x^2} + \frac{x}{t} = 0$
8. $\frac{dx}{dt} = kx(1 - \frac{x}{M})$, where k and M are positive constants
9. $\frac{dx}{dt} + x = e^{-t}x^3$
10. $tx' - 3x = tx^4$

Part B: Conceptual Questions

Answer the following questions with clear explanations and mathematical justification.

11. Explain why a Bernoulli equation with $n = 0$ or $n = 1$ reduces to a linear equation. Provide a specific example of each case.
12. Consider the general Bernoulli equation $x' + p(t)x = q(t)x^n$. Show that if $x(t) = 0$ is a solution, then this solution may not be captured by the substitution method. Under what conditions would $x(t) = 0$ be a solution?
13. For problem 8, the equation $\frac{dx}{dt} = kx(1 - \frac{x}{M})$ is known as the logistic equation. Discuss its applications in population dynamics. What do the parameters k and M represent?
14. Analyze the behavior of solutions to the Bernoulli equation $x' = ax - bx^3$ (where a and b are positive constants) for different initial conditions. What happens to the solutions as $t \to \infty$?
15. Develop a connection between Bernoulli equations and separable equations. When is a Bernoulli equation also separable? Give an example.

Differential Equations I (Math 330)

Course Notes: First-Order Applications

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Applications of First-Order Differential Equations

In this section, we explore how first-order differential equations model various real-world phenomena. These applications demonstrate the practical value of the mathematical techniques we've developed.

Application 1: Simple Population Dynamics

Population dynamics offers a natural application of first-order differential equations. Consider a population with size $P(t)$ at time t. When modeling population growth, we typically assume the population changes at a rate proportional to its current size.

Let α represent the birth rate and β represent the death rate, both expressed as constants per unit time. The net growth of the population follows:

$$\frac{dP}{dt} = \alpha P(t) - \beta P(t) = (\alpha - \beta)P(t) = rP(t)$$

Where $r = \alpha - \beta$ represents the overall growth rate constant.

This differential equation is separable:

$$\frac{dP}{dt} = rP$$
$$\frac{dP}{P} = r\,dt$$

Integrating both sides:

$$\int \frac{dP}{P} = \int r\,dt$$
$$\ln|P| = rt + C$$

Taking the exponential of both sides:

$$P = e^{rt+C} = e^C e^{rt} = P_0 e^{rt}$$

Where $P_0 = P(0)$ is the initial population.

Example (Mosquito Population)

The mosquito population in Martin increases at a rate proportional to the current population. Under ideal conditions, the population doubles each week. Initially, there are 200,000 mosquitoes, and predators consume approximately 20,000 mosquitoes per day. Find an expression for the mosquito population $P(t)$ at time t (measured in weeks).

Solution:

We can model this situation with:

$$\frac{dP}{dt} = rP(t) - \beta$$

Where:

- $rP(t)$ represents growth proportional to population
- β represents the constant predation rate
- $P(0) = 200,000$ is the initial population

First, we determine r from the information that the population doubles weekly under ideal conditions. This means:

$$\begin{aligned} P(1) &= 2P(0) \quad \text{(in the absence of predation)} \\ P_0 e^{r \cdot 1} &= 2P_0 \\ e^r &= 2 \\ r &= \ln(2) \end{aligned}$$

Next, we convert the daily predation rate to a weekly rate:

$$\begin{aligned} \beta &= 20,000 \text{ mosquitoes/day} \times 7 \text{ days/week} \\ &= 140,000 \text{ mosquitoes/week} \end{aligned}$$

Our differential equation becomes:

$$\frac{dP}{dt} = (\ln 2)P - 140,000$$

This is a linear first-order differential equation. Rearranging to standard form:

$$\frac{dP}{dt} - (\ln 2)P = -140,000$$

Using the integrating factor method with $\mu(t) = e^{-(\ln 2)t} = 2^{-t}$:

$$\begin{aligned} \frac{d}{dt}(2^{-t}P) &= -140,000 \cdot 2^{-t} \\ 2^{-t}P &= -140,000 \int 2^{-t}\, dt + C \\ &= -140,000 \cdot \frac{-1}{\ln 2} \cdot 2^{-t} + C \\ &= \frac{140,000}{\ln 2} \cdot 2^{-t} + C \end{aligned}$$

Multiplying by 2^t:

$$P(t) = \frac{140,000}{\ln 2} + C \cdot 2^t$$

Using $P(0) = 200,000$:

$$\begin{aligned} 200,000 &= \frac{140,000}{\ln 2} + C \\ C &= 200,000 - \frac{140,000}{\ln 2} \end{aligned}$$

Therefore:

$$\begin{aligned} P(t) &= \frac{140,000}{\ln 2} + \left(200,000 - \frac{140,000}{\ln 2}\right) 2^t \\ &= \frac{140,000}{\ln 2} + 200,000 \cdot 2^t - \frac{140,000}{\ln 2} \cdot 2^t \\ &= 200,000 \cdot 2^t + \frac{140,000}{\ln 2}(1 - 2^t) \end{aligned}$$

For large values of t, the term 2^t dominates, leading to exponential growth when $200,000 > \frac{140,000}{\ln 2}$, or equivalently, when the growth rate exceeds the predation rate.

Application 2: Advanced Population Dynamics

The simple exponential growth model has a significant limitation: it fails to account for environmental constraints such as limited resources, space, or food. In reality, populations cannot grow indefinitely.

To address this limitation, we introduce the logistic growth model, which incorporates a carrying capacity—the maximum sustainable population size that the environment can support.

The logistic growth model is given by:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

Where:

- r is the intrinsic growth rate (growth rate under ideal conditions)
- K is the carrying capacity
- $P(t)$ is the population at time t

This model has several important properties:

- When P is small compared to K, the term $\left(1 - \frac{P}{K}\right) \approx 1$, so $\frac{dP}{dt} \approx rP$ (approximately exponential growth)
- When $P = K$, we have $\frac{dP}{dt} = 0$ (zero growth)
- When $P > K$, the growth rate becomes negative, causing the population to decrease

To solve this differential equation, we use separation of variables:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

$$\frac{dP}{P\left(1 - \frac{P}{K}\right)} = r\,dt$$

Using partial fractions:

$$\frac{1}{P\left(1 - \frac{P}{K}\right)} = \frac{A}{P} + \frac{B}{1 - \frac{P}{K}}$$

Finding common denominators:

$$\frac{1}{P\left(1 - \frac{P}{K}\right)} = \frac{A\left(1 - \frac{P}{K}\right) + BP}{P\left(1 - \frac{P}{K}\right)}$$

This gives us:

$$1 = A\left(1 - \frac{P}{K}\right) + BP$$

$$1 = A - \frac{AP}{K} + BP$$

Comparing coefficients:

$$\text{Constant terms:} \quad 1 = A$$

$$\text{Terms with } P\text{:} \quad 0 = -\frac{A}{K} + B = -\frac{1}{K} + B$$

Therefore, $A = 1$ and $B = \frac{1}{K}$, giving:

$$\begin{aligned}
\frac{1}{P\left(1 - \frac{P}{K}\right)} &= \frac{1}{P} + \frac{1}{K}\frac{1}{1 - \frac{P}{K}} \\
&= \frac{1}{P} + \frac{1}{K - P}
\end{aligned}$$

Returning to our differential equation:

$$\begin{aligned}
\frac{dP}{P\left(1 - \frac{P}{K}\right)} &= r\,dt \\
\left(\frac{1}{P} + \frac{1}{K - P}\right) dP &= r\,dt
\end{aligned}$$

Integrating both sides:

$$\begin{aligned}
\int \left(\frac{1}{P} + \frac{1}{K - P}\right) dP &= \int r\,dt \\
\ln|P| - \ln|K - P| &= rt + C_1 \\
\ln\left|\frac{P}{K - P}\right| &= rt + C_1
\end{aligned}$$

Taking the exponential of both sides:

$$\begin{aligned}
\frac{P}{K - P} &= e^{rt + C_1} = C_2 e^{rt} \\
P &= (K - P)C_2 e^{rt} \\
P &= KC_2 e^{rt} - PC_2 e^{rt} \\
P(1 + C_2 e^{rt}) &= KC_2 e^{rt} \\
P &= \frac{KC_2 e^{rt}}{1 + C_2 e^{rt}}
\end{aligned}$$

If we let $C_2 = \frac{P_0}{K - P_0}$, where $P_0 = P(0)$, then:

$$P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}$$

This solution produces the characteristic S-shaped curve associated with logistic growth.

Application 3: Newton's Law of Cooling

Newton's Law of Cooling describes the temperature change of an object when placed in an environment with a different temperature. The law states that the rate of temperature change is proportional to the difference between the object's temperature and the ambient temperature.

Mathematically:

$$\frac{dT}{dt} = k(A - T(t))$$

Where:

- $T(t)$ is the temperature of the object at time t
- A is the ambient temperature
- k is a positive constant that depends on the object's properties

This is a linear first-order differential equation that we can solve using separation of variables:

$$\frac{dT}{dt} = k(A - T)$$
$$\frac{dT}{A - T} = k\,dt$$

Integrating both sides:

$$\int \frac{dT}{A - T} = \int k\,dt$$
$$-\ln|A - T| = kt + C_1$$
$$\ln|A - T| = -kt + C_1$$

Taking the exponential of both sides:

$$|A - T| = e^{-kt+C_1} = e^{C_1}e^{-kt} = C_2e^{-kt}$$

Since typically $A - T$ starts positive and approaches zero from above, we can remove the absolute value signs:

$$A - T = C_2e^{-kt}$$
$$T = A - C_2e^{-kt}$$

Using the initial condition $T(0) = T_0$:

$$T_0 = A - C_2$$
$$C_2 = A - T_0$$

Therefore:

$$T(t) = A - (A - T_0)e^{-kt} = A + (T_0 - A)e^{-kt}$$

Example (Coffee Cooling)

A cup of coffee has an initial temperature of 210°F and is placed in a room with a constant temperature of 70°F. After 5 minutes, the coffee temperature has dropped to 185°F. When will the coffee reach a drinkable temperature of 160°F?

Solution:

Using Newton's Law of Cooling:

$$\frac{dT}{dt} = k(A - T)$$

With:

- Initial temperature: $T(0) = 210F$
- Ambient temperature: $A = 70F$
- $T(5) = 185F$

The general solution is:

$$T(t) = A + (T_0 - A)e^{-kt}$$

Substituting our values:

$$T(t) = 70 + (210 - 70)e^{-kt}$$
$$= 70 + 140e^{-kt}$$

To find k, we use the condition $T(5) = 185F$:

$$\begin{aligned}
185 &= 70 + 140e^{-5k} \\
115 &= 140e^{-5k} \\
\frac{115}{140} &= e^{-5k} \\
\ln\left(\frac{115}{140}\right) &= -5k \\
k &= -\frac{1}{5}\ln\left(\frac{115}{140}\right) = \frac{1}{5}\ln\left(\frac{140}{115}\right)
\end{aligned}$$

Now we can find when $T(t) = 160F$:

$$\begin{aligned}
160 &= 70 + 140e^{-kt} \\
90 &= 140e^{-kt} \\
\frac{90}{140} &= e^{-kt} \\
\ln\left(\frac{90}{140}\right) &= -kt \\
t &= -\frac{1}{k}\ln\left(\frac{90}{140}\right) = \frac{1}{k}\ln\left(\frac{140}{90}\right) \\
&= 5\frac{\ln\left(\frac{140}{90}\right)}{\ln\left(\frac{140}{115}\right)}
\end{aligned}$$

Evaluating this expression gives approximately $t \approx 9.07$ minutes.
Therefore, the coffee will reach 160°F after about 9 minutes and 4 seconds.

Application 4: Single-compartment Mixing Problems

Mixing problems involve tracking the amount of a substance (solute) in a container where fluid flows in and out. These problems are common in chemistry, environmental science, and engineering.

The fundamental principle is conservation of mass:

Rate of change of solute = Rate of solute flowing in - Rate of solute flowing out

Mathematically, if $x(t)$ represents the amount of solute in the container at time t:

$$\frac{dx}{dt} = r_{in}c_{in} - r_{out}c_{out}$$

Where:

- r_{in} is the flow rate into the container (volume/time)
- c_{in} is the concentration of solute in the incoming fluid (amount/volume)
- r_{out} is the flow rate out of the container (volume/time)
- c_{out} is the concentration of solute in the outgoing fluid (amount/volume)

Assuming the mixture is well-stirred, the outgoing concentration equals the concentration in the tank:

$$c_{out} = \frac{x(t)}{V(t)}$$

Where $V(t)$ is the volume of fluid in the container at time t.

Example (Salt Concentration)

Consider a fish tank that initially contains 150 liters of water with 20 grams of dissolved salt. The salt concentration needs to be increased from $\frac{20}{150}$ grams per liter to 1 gram per liter. Water containing 3 grams of salt per liter flows into the tank at 2 liters per minute, and the well-mixed solution flows out at the same rate. Find the time required to reach the target concentration.

Solution:

Let $x(t)$ be the amount of salt (in grams) in the tank at time t.

Using conservation of mass:

$$\begin{aligned}\frac{dx}{dt} &= r_{in}c_{in} - r_{out}c_{out} \\ &= 2 \text{ L/min} \times 3 \text{ g/L} - 2 \text{ L/min} \times \frac{x(t)}{150 \text{ L}} \\ &= 6 - \frac{2x(t)}{150} \\ &= 6 - \frac{x(t)}{75}\end{aligned}$$

This is a linear first-order differential equation:

$$\frac{dx}{dt} + \frac{x}{75} = 6$$

Using the integrating factor method with $\mu(t) = e^{\int \frac{1}{75} dt} = e^{\frac{t}{75}}$:

$$\begin{aligned}\frac{d}{dt}\left(e^{\frac{t}{75}}x\right) &= 6e^{\frac{t}{75}} \\ e^{\frac{t}{75}}x &= \int 6e^{\frac{t}{75}}\,dt \\ &= 6 \times 75 \times e^{\frac{t}{75}} + C \\ &= 450e^{\frac{t}{75}} + C\end{aligned}$$

Therefore:

$$x(t) = 450 + Ce^{-\frac{t}{75}}$$

Using the initial condition $x(0) = 20$:

$$\begin{aligned}20 &= 450 + C \\ C &= 20 - 450 = -430\end{aligned}$$

So:

$$x(t) = 450 - 430e^{-\frac{t}{75}}$$

We want to find when the concentration reaches 1 gram per liter:

$$\begin{aligned}\frac{x(t)}{150} &= 1 \\ x(t) &= 150\end{aligned}$$

Substituting:

$$\begin{aligned}
150 &= 450 - 430e^{-\frac{t}{75}} \\
-300 &= -430e^{-\frac{t}{75}} \\
\frac{300}{430} &= e^{-\frac{t}{75}} \\
\ln\left(\frac{300}{430}\right) &= -\frac{t}{75} \\
t &= -75\ln\left(\frac{300}{430}\right) = 75\ln\left(\frac{430}{300}\right)
\end{aligned}$$

Calculating this gives approximately $t \approx 28.83$ minutes.
Therefore, it will take about 28 minutes and 50 seconds to reach the desired salt concentration.

Homework Assignment

Solve the following application problems.

1. Show how the general solution of Newton's Law of Cooling differential equation is derived. Show all steps from the differential equation $\frac{dT}{dt} = k(A - T(t))$ to the general solution $T(t) = A - (A - T_0)e^{-kt}$.

2. Suppose a population satisfies the differential equation $\frac{dx}{dt} = (0.5 + \sin t)x$:
   (a) If $x(0) = 1$, find the time at which the population doubles.
   (b) Choose another initial condition and determine whether the doubling time depends on the initial population.
   (c) If the growth rate is replaced by its average value 0.5, determine the doubling time.
   (d) If the term $\sin t$ is replaced by $\sin(2\pi t)$, what effect does this have on the doubling time?

3. An ice-cold drink at 35°F is taken out of a 35°F refrigerator and placed in a room at 75°F. After 10 minutes, the drink has warmed to 45°F. What will the temperature be after 30 minutes? Create a graph showing the temperature over time.

4. Consider a tank containing 200 liters of water with 50 grams of dissolved salt. The salt concentration needs to be decreased to 0.1 grams per liter. Fresh water flows into the tank at 2 liters per minute, and the mixture flows out at the same rate. How long will it take to reach the target concentration?

5. A tank initially contains 200 liters of water with 50 grams of salt. Fresh water flows in at 1.5 liters per minute, but due to a leak, water drains at 2 liters per minute. Will the concentration ever reach 0.1 grams per liter before the tank empties? If so, when?

Differential Equations I (Math 330)

Course Notes: Introduction to Second-order Differential Equations

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Introduction to Second-order Differential Equations

We now examine differential equations expressed as:

$$x''(t) = F(t, x, x')$$

where F is some function. These are known as second-order differential equations.

Key Definitions

Linear Equation: A second-order differential equation is considered linear if it can be written in the standard form:

$$x'' + p(t)x' + q(t)x = f(t)$$

where p, q, and f are functions of t.

Homogeneous Linear Equation: A linear equation is homogeneous if $f(t) = 0$, which gives us:

$$x'' + p(t)x' + q(t)x = 0$$

General Theory of Homogeneous Linear Equations

For this section, we'll primarily focus on linear second-order equations of the form:

$$x'' + p(t)x' + q(t)x = f(t)$$

We typically approach the solution by breaking it into two components, similar to how we would split the function $f(t)$ into $f(t) + 0$.

Important Definitions

Associated Homogeneous Equation: For any linear second-order equation, there is an associated homogeneous equation:

$$x'' + p(t)x' + q(t)x = 0$$

The general solution to this equation is denoted as x_h and referred to as the homogeneous solution.

Particular Solution: There is also a particular solution x_p to the complete equation:

$$x'' + p(t)x' + q(t)x = f(t)$$

The Superposition Principle

Theorem (Superposition Principle):
If x_h is a general solution of:

$$x'' + p(t)x' + q(t)x = 0$$

and x_p is any solution of:

$$x'' + p(t)x' + q(t)x = f(t)$$

then the sum $x = x_h + x_p$ is a general solution of:

$$x'' + p(t)x' + q(t)x = f(t)$$

Lemma:
If x_1 and x_2 are both solutions of:

$$x'' + p(t)x' + q(t)x = f(t)$$

then $Ax_1 + Bx_2$ is also a solution for any constants A and B.

Example 1

Let's verify that $x_1(t) = e^t$ and $x_2(t) = e^{2t}$ are both solutions of the differential equation:

$$x'' - 3x' + 2x = 0$$

And then confirm that their linear combination $x = Ax_1 + Bx_2$ is also a solution.
For $x_1 = e^t$:

$$\begin{aligned} x_1' &= e^t \\ x_1'' &= e^t \\ x_1'' - 3x_1' + 2x_1 &= e^t - 3e^t + 2e^t = 0\checkmark \end{aligned}$$

For $x_2 = e^{2t}$:

$$\begin{aligned} x_2' &= 2e^{2t} \\ x_2'' &= 4e^{2t} \\ x_2'' - 3x_2' + 2x_2 &= 4e^{2t} - 3(2e^{2t}) + 2e^{2t} = 0\checkmark \end{aligned}$$

For the linear combination $x = Ae^t + Be^{2t}$:

$$\begin{aligned} x' &= Ae^t + 2Be^{2t} \\ x'' &= Ae^t + 4Be^{2t} \\ x'' - 3x' + 2x &= (Ae^t + 4Be^{2t}) - 3(Ae^t + 2Be^{2t}) + 2(Ae^t + Be^{2t}) \\ &= (A - 3A + 2A)e^t + (4B - 3(2B) + 2B)e^{2t} = 0 \end{aligned}$$

Finding Solutions and the Wronskian

How can we systematically find solutions like x_1 and x_2 from the previous example? And how do we know when we've found all the necessary solutions?

Definition (Wronskian):
If x_1 and x_2 are solutions of the second-order linear homogeneous equation $x'' + p(t)x' + q(t)x = 0$, then the determinant:

$$W(x_1, x_2)(t) \equiv \det \begin{pmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{pmatrix} \equiv x_1(t)x_2'(t) - x_2(t)x_1'(t)$$

is called the Wronskian of the functions x_1 and x_2.
(Note: The concept of a Wronskian can be extended to higher-order equations.)

Theorem:
If x_1 and x_2 are two solutions of $x'' + px' + q = 0$, and their Wronskian $x_1x_2' - x_2x_1'$ is non-zero for all values of t, then $x = C_1x_1 + C_2x_2$ represents a general solution of $x'' + px' + q = 0$.

Definition (Fundamental Solution Set):
A pair of solutions $\{x_1, x_2\}$ of the equation $x'' + px' + q = 0$ satisfying $W(x_1, x_2) \neq 0$ for all values of t in an interval where the coefficient functions are continuous, is called a fundamental solution set in that interval.

Example 2

Let's verify that $\{e^t, e^{2t}\}$ forms a fundamental solution set for the differential equation $x'' - 3x' + 2x = 0$.

$$\begin{aligned} W(e^t, e^{2t}) &= \det \begin{pmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{pmatrix} \\ &= e^t \cdot 2e^{2t} - e^{2t} \cdot e^t \\ &= 2e^{3t} - e^{3t} \\ &= e^{3t} \end{aligned}$$

Since $e^{3t} \neq 0$ for all $t \in (-\infty, \infty)$, we can confirm that $\{e^t, e^{2t}\}$ is indeed a fundamental solution set for the differential equation across all real numbers.

Practice Examples

Determine whether the given pairs of functions form a fundamental solution set for some linear second-order equation by computing the Wronskian and applying the appropriate theorem.

1. $x_1(t) = t^2 + 5t$, $x_2(t) = t^2 - 5t$
2. $x_1(t) = \cos(3t)$, $x_2(t) = 4\cos^3(t) - 3\cos t$
3. $x_1(t) = e^{3t}$, $x_2(t) = e^{3(t-1)}$
4. $x_1(t) = t$, $x_2(t) = t^{-1}$
5. $x_1(t) = \cos t$, $x_2(t) = \sin(2t)$

Homework Assignment

Determine whether each of the following equations is linear or nonlinear in the dependent variable x. If it is linear, state whether or not it is homogeneous.

1. $x'' + tx' + t^2 x = 0$
2. $x'' + t^2 x' + tx^2 = 2t + 1$
3. $x'' + \sin(t)x' - \cos(t)x = \pi$
4. $x'' - 2x' + \sin(t) = 1$
5. $x'' - xx' + 2t = 0$

Two functions x_1 and x_2 are given. Compute the Wronskian, $W(x_1, x_2)$, and determine whether or not $\{x_1, x_2\}$ form a fundamental solution set for some linear second-order differential equation.

6. $x_1(t) = 3t - 5$, $x_2(t) = 9t - 15$
7. $x_1(t) = \cos t$, $x_2(t) = \sin t$
8. $x_1(t) = e^{r_1 t}$, $x_2(t) = e^{r_2 t}$, where $r_1 \neq r_2$
9. $x_1(t) = e^{2t}\cos(3t)$, $x_2(t) = e^{2t}\sin(3t)$
10. $x_1(t) = e^{\lambda t}\cos(\mu t)$, $x_2(t) = e^{\lambda t}\sin(\mu t)$. Are there any restrictions that need to be placed on λ or μ in order to guarantee a fundamental solution set? (think similar to the restriction imposed on r_1, r_2 in problem 8)

Differential Equations I (Math 330)

Course Notes: Homogeneous Linear Second-order Differential Equations with Constant Coefficients

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Homogeneous Linear Second-order Differential Equations with Constant Coefficients

We now consider differential equations of the form:

$$ax''(t) + bx'(t) + cx(t) = 0 \tag{1}$$

for constants $a \neq 0$, b, and c.

What functions seem reasonable when considering constant multiples of the function and its derivatives are to be added and should cancel out to zero?

Examining Polynomial Functions

Let's consider a specific quadratic function, $x(t) = t^2 + t + 1$:

$$x(t) = t^2 + t + 1 \tag{2}$$
$$x'(t) = 2t + 1 \tag{3}$$
$$x''(t) = 2 \tag{4}$$

Substituting into the differential equation:

$$ax'' + bx' + cx = a(2) + b(2t + 1) + c(t^2 + t + 1) \tag{5}$$
$$= (c)t^2 + (2b + c)t + (2a + b + c) \tag{6}$$

For this to equal zero for all t, we need:

$$\begin{cases} c = 0 \\ 2b + c = 0 \\ 2a + b + c = 0 \end{cases} \tag{7}$$

This system yields only the trivial solution $a = b = c = 0$, which is no longer a second-order differential equation. This suggests polynomials aren't generally solutions.

Examining Exponential Functions

Let's try a general exponential function, $x(t) = e^{rt}$:

$$x(t) = e^{rt} \tag{8}$$
$$x'(t) = re^{rt} \tag{9}$$
$$x''(t) = r^2e^{rt} \tag{10}$$

Substituting into the differential equation:

$$ax'' + bx' + cx = a(r^2e^{rt}) + b(re^{rt}) + c(e^{rt}) \tag{11}$$
$$= e^{rt}(ar^2 + br + c) \tag{12}$$

For this to equal zero, we need $ar^2 + br + c = 0$. This is a quadratic equation in r that can typically give us two values of r that yield a fundamental solution set.

The Characteristic Polynomial

Definition (Characteristic Polynomial): The polynomial $P(r) = ar^2 + br + c$ is called the characteristic polynomial of the differential equation:

$$ax''(t) + bx'(t) + cx(t) = 0 \tag{13}$$

Our goal is to find the roots of this polynomial and translate them into solutions of the differential equation. We classify the solutions based on the discriminant $K = b^2 - 4ac$.

Case 1: $K > 0$ (Distinct Real Roots)

When $K > 0$, the characteristic polynomial yields two distinct real roots, r_1 and r_2, where $r_1 \neq r_2$.

The exponential functions $x_1(t) = e^{r_1 t}$ and $x_2(t) = e^{r_2 t}$ are two solutions of the differential equation.

To verify they form a fundamental solution set, we compute the Wronskian:

$$W(e^{r_1 t}, e^{r_2 t}) = \det \begin{pmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{pmatrix} \tag{14}$$

$$= e^{r_1 t} \cdot r_2 e^{r_2 t} - e^{r_2 t} \cdot r_1 e^{r_1 t} \tag{15}$$

$$= (r_2 - r_1) e^{(r_1 + r_2)t} \tag{16}$$

Since $r_1 \neq r_2$, we have $W \neq 0$ for all t, confirming these form a fundamental solution set.

The general solution to the differential equation is:

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{17}$$

where C_1 and C_2 are arbitrary constants.

Case 2: $K = 0$ (Repeated Real Root)

When $K = 0$, the characteristic polynomial yields one repeated real root $\bar{r}$, where $\bar{r} = -\frac{b}{2a}$ or $2a\bar{r} + b = 0$.

The exponential function $x_1(t) = e^{\bar{r}t}$ is one solution, but we need a second linearly independent solution.

Let's try $x_2(t) = te^{\bar{r}t}$:

$$x_2(t) = te^{\bar{r}t} \tag{18}$$

$$x_2'(t) = e^{\bar{r}t} + \bar{r}te^{\bar{r}t} \tag{19}$$

$$x_2''(t) = 2\bar{r}e^{\bar{r}t} + \bar{r}^2 te^{\bar{r}t} \tag{20}$$

Substituting into the differential equation:

$$ax_2'' + bx_2' + cx_2 = a(2\bar{r}e^{\bar{r}t} + \bar{r}^2 te^{\bar{r}t}) + b(e^{\bar{r}t} + \bar{r}te^{\bar{r}t}) + c(te^{\bar{r}t}) \tag{21}$$

$$= e^{\bar{r}t}(2a\bar{r} + b) + te^{\bar{r}t}(a\bar{r}^2 + b\bar{r} + c) \tag{22}$$

Since $2a\bar{r} + b = 0$ and $a\bar{r}^2 + b\bar{r} + c = 0$ (as $\bar{r}$ is a root of the characteristic equation), this expression equals zero, confirming $x_2(t) = te^{\bar{r}t}$ is also a solution.

To verify these form a fundamental solution set, we compute the Wronskian:

$$W(e^{\bar{r}t}, te^{\bar{r}t}) = \det \begin{pmatrix} e^{\bar{r}t} & te^{\bar{r}t} \\ \bar{r}e^{\bar{r}t} & e^{\bar{r}t} + \bar{r}te^{\bar{r}t} \end{pmatrix} \tag{23}$$

$$= e^{\bar{r}t} \cdot (e^{\bar{r}t} + \bar{r}te^{\bar{r}t}) - te^{\bar{r}t} \cdot \bar{r}e^{\bar{r}t} \tag{24}$$

$$= e^{2\bar{r}t} + \bar{r}te^{2\bar{r}t} - t\bar{r}e^{2\bar{r}t} \tag{25}$$

$$= e^{2\bar{r}t} \tag{26}$$

Since $e^{2\bar{r}t} \neq 0$ for all t, these functions form a fundamental solution set.

The general solution to the differential equation is:

$$x(t) = C_1 e^{\bar{r}t} + C_2 te^{\bar{r}t} \tag{27}$$

where C_1 and C_2 are arbitrary constants.

Case 3: $K < 0$ (Complex Conjugate Roots)

When $K < 0$, the characteristic polynomial yields a complex conjugate pair of roots, $\alpha \pm \beta i$.

The complex exponential functions $z_1(t) = e^{(\alpha+\beta i)t}$ and $z_2(t) = e^{(\alpha-\beta i)t}$ are solutions to the differential equation.

To find real solutions, we use complex analysis to obtain:

$$x_1(t) = e^{\alpha t}\cos(\beta t) \tag{28}$$
$$x_2(t) = e^{\alpha t}\sin(\beta t) \tag{29}$$

We can verify these form a fundamental solution set by computing the Wronskian:

$$\begin{aligned} W(e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)) &= e^{\alpha t}\cos(\beta t)\cdot(e^{\alpha t}\beta\cos(\beta t) + \alpha e^{\alpha t}\sin(\beta t)) & (30)\\ &\quad - e^{\alpha t}\sin(\beta t)\cdot(-e^{\alpha t}\beta\sin(\beta t) + \alpha e^{\alpha t}\cos(\beta t)) & (31)\\ &= \beta e^{2\alpha t} & (32) \end{aligned}$$

Since $\beta \neq 0$ (as the roots are complex), $W \neq 0$ for all t, confirming these functions form a fundamental solution set.

The general solution to the differential equation is:

$$x(t) = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t) \tag{33}$$

where C_1 and C_2 are arbitrary constants.

Examples

Find the general solution to each of the following differential equations:

1. $2x'' - 6x' + 4x = 0$
2. $2x'' - 3x = 0$
3. $x'' - 6x' + 9x = 0$
4. $x'' + 2x' - x = 0$
5. $36x'' + 12x' + 37x = 0$
6. $x'' + 4x = 0$
7. $x'' - 2x' - 2x = 0$

Equations of Order Greater Than Two

Consider the general higher-order equation:

$$a_n x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1 x' + a_0 x = 0 \tag{34}$$

This is a homogeneous linear equation with constant coefficients. To solve it, we again use the characteristic polynomial:

$$P(r) = a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 \tag{35}$$

Assuming the coefficients are real numbers, $P(r)$ has exactly n roots and can be factored into linear and irreducible quadratic factors. Linear factors correspond to real roots and exponential solutions. Irreducible quadratic factors correspond to complex conjugate pairs of roots and exponential-trigonometric solutions.

The multiplicity of a root corresponds to the "degree minus 1" of the coefficient of the solution, and both real and complex roots can have multiplicities.

Examples

Find the general solution to the differential equations having the following characteristic polynomials:

1. $P(r) = (r-1)(r+2)(r-3)(r+4)$
2. $P(r) = r^3(r^2+1)(r^2-9)$

Find the general solution to each of the following differential equations:

7. $x^{(4)} - x^{(3)} - x'' - x' - 2x = 0$
8. $x''' - 3x'' - 13x' + 15x = 0$

Homework Assignment

Find the general solution to the differential equations having the following characteristic polynomials:

1. $P(r) = r(r-1)^2(r^2+3)^4$
2. $P(r) = r^2(r-11)^3$
3. $P(r) = (r^2-4r+13)^2(r^2-6r+34)$
4. $P(r) = r^8 - 2r^7 + r^6 - r^4 + 2r^3 - r^2$

Find the general solution to each of the following differential equations.

1. $-x'' + 4x' = 0$
2. $x'' + 6x' + 9x = 0$
3. $-3x'' + 3x' - 2x = 0$
4. $3x'' - 8x' + 2x = 0$
5. $2x^{(3)} - 13x'' + 24x' - 9x = 0$
6. $21x^{(4)} + 22x^{(3)} - 99x'' - 72x' + 28x = 0$
7. $x^{(5)} - 5x^{(4)} - 5x^{(3)} + 25x'' + 40x' + 16x = 0$
8. $x^{(6)} + 12x^{(4)} + 48x^{(2)} + 64x = 0$
9. $2x^{(4)} + x^{(3)} - 35x'' - 113x' + 65x = 0$
10. $x^{(4)} + 3x^{(3)} - 19x'' + 27x' - 252x = 0$

Differential Equations I (Math 330)

Course Notes: Method of Undetermined Coefficients

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Method of Undetermined Coefficients

This section examines differential equations in the form:

$$ax''(t) + bx'(t) + cx(t) = f(t)$$

Where $a \neq 0$, b, and c are constants, and $f(t)$ takes specific forms. This technique can be extended to higher-order equations with minimal modifications.

Example 1

Let's analyze the differential equation:

$$x'' + 4x' + 4x = 2e^{-3t}$$

Applying the Superposition Principle discussed previously, we can approach this as:

$$x'' + 4x' + 4x = 0 + 2e^{-3t}$$

First, we need the homogeneous solution $x_h(t)$ by solving:

$$x'' + 4x' + 4x = 0$$

Using our techniques from last week, we find $x_h(t) = C_1e^{-2t} + C_2te^{-2t}$.
Next, we need to determine a particular solution $x_p(t)$ to:

$$x'' + 4x' + 4x = 2e^{-3t}$$

For the particular solution, what form should we guess? Since the right side contains e^{-3t}, a reasonable guess would be:

$$x_p(t) = Ae^{-3t}$$

Let's verify this works by finding the specific value of coefficient A:

$$\begin{aligned} x_p'(t) &= -3Ae^{-3t} \\ x_p''(t) &= 9Ae^{-3t} \end{aligned}$$

Substituting into our equation:

$$\begin{aligned} x'' + 4x' + 4x &= 9Ae^{-3t} + 4(-3Ae^{-3t}) + 4(Ae^{-3t}) \\ &= (9A - 12A + 4A)e^{-3t} \\ &= Ae^{-3t} \end{aligned}$$

Since we need $Ae^{-3t} = 2e^{-3t}$, we get $A = 2$.
Therefore, $x_p(t) = 2e^{-3t}$.
The general solution is:

$$x(t) = x_h(t) + x_p(t) = C_1e^{-2t} + C_2te^{-2t} + 2e^{-3t}$$

Example 2

Consider the differential equation:

$$x'' + 4x' + 4x = 2\cos t$$

We again use the Superposition Principle with the same homogeneous solution. For the particular solution, let's try:

$$x_p(t) = A\cos t$$

Taking derivatives:

$$x_p'(t) = -A\sin t$$
$$x_p''(t) = -A\cos t$$

Substituting:

$$\begin{aligned} x'' + 4x' + 4x &= (-A\cos t) + 4(-A\sin t) + 4(A\cos t) \\ &= -4A\sin t + 3A\cos t \end{aligned}$$

This doesn't match our target $2\cos t$ since we need both $-4A\sin t = 0$ and $3A\cos t = 2\cos t$, which can't be satisfied simultaneously. Our guess was insufficient.

Let's try a more general form:

$$x_p(t) = A\cos t + B\sin t$$

Taking derivatives:

$$x_p'(t) = -A\sin t + B\cos t$$
$$x_p''(t) = -A\cos t - B\sin t$$

Substituting:

$$\begin{aligned} x'' + 4x' + 4x &= (-A\cos t - B\sin t) + 4(-A\sin t + B\cos t) + 4(A\cos t + B\sin t) \\ &= (-A + 4B + 4A)\cos t + (-B - 4A + 4B)\sin t \\ &= (3A + 4B)\cos t + (-4A + 3B)\sin t \end{aligned}$$

For this to equal $2\cos t$, we need:

$$3A + 4B = 2$$
$$-4A + 3B = 0$$

Solving this system:

$$\begin{aligned} -4A + 3B = 0 &\implies B = \frac{4A}{3} \\ 3A + 4\left(\frac{4A}{3}\right) &= 2 \\ 3A + \frac{16A}{3} &= 2 \\ \frac{9A + 16A}{3} &= 2 \\ \frac{25A}{3} &= 2 \end{aligned}$$

Therefore $A = \frac{6}{25}$ and $B = \frac{8}{25}$.

The general solution is:

$$x(t) = C_1 e^{-2t} + C_2 t e^{-2t} + \frac{6}{25}\cos t + \frac{8}{25}\sin t$$

Example 3

Now consider:

$$x'' + 4x' + 4x = e^{-2t}$$

This case requires special attention! Notice that e^{-2t} is already a solution to the homogeneous equation, so our usual guess $x_p(t) = Ae^{-2t}$ won't work.

When the forcing function matches a solution of the homogeneous equation, we multiply by t to create a new form:

$$x_p(t) = Ate^{-2t}$$

However, te^{-2t} is also a solution to the homogeneous equation, so we need to multiply by t again:

$$x_p(t) = At^2e^{-2t}$$

Let's verify this works:

$$\begin{aligned}
x_p'(t) &= 2Ate^{-2t} - 2At^2e^{-2t} = 2At(1-t)e^{-2t} \\
x_p''(t) &= 2A(1-t)e^{-2t} + 2At(1-t)(-2)e^{-2t} + 2A(-1)e^{-2t} \\
&= 2A(1-t)e^{-2t} - 4At(1-t)e^{-2t} - 2Ae^{-2t} \\
&= 2Ae^{-2t} - 2Ate^{-2t} - 4At(1-t)e^{-2t} - 2Ae^{-2t} \\
&= -2Ate^{-2t} - 4At(1-t)e^{-2t}
\end{aligned}$$

Substituting into our equation:

$$x'' + 4x' + 4x = [-2Ate^{-2t} - 4At(1-t)e^{-2t}] + 4[2At(1-t)e^{-2t}] + 4[At^2e^{-2t}]$$

After simplification (which involves some algebra), we get:

$$x'' + 4x' + 4x = 2Ae^{-2t}$$

Since we need $2Ae^{-2t} = e^{-2t}$, we find $A = \frac{1}{2}$.

The general solution is:

$$x(t) = C_1e^{-2t} + C_2te^{-2t} + \frac{1}{2}t^2e^{-2t}$$

Example 4

What happens when we combine forcing functions?

$$x'' + 4x' + 4x = 2e^{-3t} + 2\cos t + e^{-2t}$$

For this type of equation, we can find the particular solution for each term separately and add them together:

1. For $2e^{-3t}$: $x_{p1}(t) = 2e^{-3t}$
2. For $2\cos t$: $x_{p2}(t) = \frac{6}{25}\cos t + \frac{8}{25}\sin t$
3. For e^{-2t}: $x_{p3}(t) = \frac{1}{2}t^2e^{-2t}$

The complete general solution is:

$$x(t) = C_1e^{-2t} + C_2te^{-2t} + 2e^{-3t} + \frac{6}{25}\cos t + \frac{8}{25}\sin t + \frac{1}{2}t^2e^{-2t}$$

Method of Undetermined Coefficients: Quick Reference

Function, $f(t)$	Form of x_p*	Sample $f(t)$	x_p used
Exponential ae^{bt}	Ae^{bt}	$5e^{-2t}$	Ae^{-2t}
Polynomial (degree n)	$A_0 + A_1 t + \cdots + A_n t^n$	$1 + 2t - 7t^3$	$A + Bt + Ct^2 + Dt^3$
$\alpha \sin(bt) + \beta \cos(bt)$	$A \sin(bt) + B \cos(bt)$	$-2\cos(5t)$	$A\cos(5t) + B\sin(5t)$

*Note: If x_p is a solution of the associated homogeneous equation, then use $t^k x_p$ where k is the smallest positive integer such that $t^k x_p$ is not a homogeneous solution.

Are there other function types missing from this table that we should consider?

Additional Examples

Find the general solution to the following differential equations:

1. $x'' - 2x' - 3x = 3e^{2t}$
2. $x'' - 2x' - 3x = -3te^{-t}$
3. $x'' + x' + 4x = 2\sinh t$, Hint: $\sinh t = \frac{e^t - e^{-t}}{2}$

Find the solution to the following initial value problems:

7. $x'' + x' - 2x = 2t$, $x(0) = 0$, $x'(0) = 1$
8. $x'' - 2x' + x = te^t + 4$, $x(0) = 1$, $x'(0) = 1$

Example 5

Let's examine a higher-order differential equation:

$$x^{(4)} - 3x'' - 8x = \sin t$$

First, we need the homogeneous solution by solving:

$$x^{(4)} - 3x'' - 8x = 0$$

The characteristic equation is $r^4 - 3r^2 - 8 = 0$, which gives us:

$$x_h(t) = C_1 e^{\sqrt{\frac{3+\sqrt{41}}{2}}t} + C_2 e^{-\sqrt{\frac{3+\sqrt{41}}{2}}t} + C_3 e^{\sqrt{\frac{3-\sqrt{41}}{2}}t} + C_4 e^{-\sqrt{\frac{3-\sqrt{41}}{2}}t}$$

Since i (from $\sin t = \operatorname{Im}(e^{it})$) doesn't match any roots of the characteristic polynomial, our particular solution takes the form:

$$x_p(t) = A\cos t + B\sin t$$

After substituting into the equation and solving for coefficients, the general solution is:

$$x(t) = C_1 e^{\sqrt{\frac{3+\sqrt{41}}{2}}t} + C_2 e^{-\sqrt{\frac{3+\sqrt{41}}{2}}t} + C_3 e^{\sqrt{\frac{3-\sqrt{41}}{2}}t} + C_4 e^{-\sqrt{\frac{3-\sqrt{41}}{2}}t} + A\cos t + B\sin t$$

where A and B are the determined coefficients.

Homework Assignements

Find the general solution to the following differential equations:

1. $x'' + 2x' + 5x = 3\sin(2t)$
2. $x'' + 2x' = 3 + 4\sin(2t)$
3. $x'' + 2x' + x = 2e^{-t}$
4. $x'' + 9x = t^2e^{3t} + 6$
5. $2x'' + 3x' + x = t^2 + 3\sin t$
6. $x'' + x = 3\sin(2t) + t\cos(2t)$
7. $x'' + \omega_0^2 x = \cos(\omega_0 t)$
8. $x'' - x' - 2x = \cosh(2t)$, HINT: $\cosh t = \frac{e^t+e^{-t}}{2}$
9. $x''' - x'' + 2x = \sin t$
10. $2x''' + 3x'' + x' - 6x = e^{-t}$
11. $x''' + x'' - 2x = te^t$

Find the solution to the following initial value problems:

1. $x' - x = 1$, $x(0) = 0$
2. $x'' + x = 2e^{-t}$, $x(0) = 0$, $x'(0) = 0$
3. $x'' - x' - 2x = \cos t - \sin(2t)$, $x(0) = -\frac{7}{20}$, $x'(0) = \frac{1}{5}$
4. $x'' - x = \sin t - e^{2t}$, $x(0) = 1$, $x'(0) = -1$
5. $x'' + 4x = 3\sin(2t)$, $x(0) = 2$, $x'(0) = -1$

Differential Equations I (Math 330)

Course Notes: Method of Variation of Parameters

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Method of Variation of Parameters

This technique addresses differential equations of the form:

$$x''(t) + p(t)x'(t) + q(t)x(t) = f(t)$$

Where p, q, and f are functions of t. While this approach can handle the same problems as the Method of Undetermined Coefficients from the previous section, it's more versatile and applies to a broader range of forcing functions $f(t)$. The method extends to higher-order equations, though that requires additional linear algebra background, so we'll focus on second-order equations here.

General Method

We begin with the general solution to the homogeneous equation (which is straightforward for constant coefficients and can be computed numerically otherwise):

$$x_h(t) = C_1x_1(t) + C_2x_2(t)$$

For convenience, we'll abbreviate this as:

$$x_h = C_1x_1 + C_2x_2$$

Where $\{x_1, x_2\}$ forms a fundamental solution set for the homogeneous equation (when $f(t) \equiv 0$). The key insight of this method is to assume the particular solution has the form:

$$x_p = v_1x_1 + v_2x_2$$

Where v_1 and v_2 are non-constant functions of t that we need to determine. Since we have two unknown functions, we need two conditions they must satisfy.

Condition 1:

Let's examine the derivative of x_p using the product rule:

$$x_p' = v_1x_1' + v_1'x_1 + v_2x_2' + v_2'x_2$$

To simplify our work with x_p' and x_p'', we impose the first condition:

$$v_1'x_1 + v_2'x_2 \equiv 0$$

This simplifies the derivative to:

$$\boxed{x_p' = v_1x_1' + v_2x_2'}$$

Condition 2:

Now we find x_p'' based on our first condition, again using the product rule:

$$x_p'' = v_1x_1'' + v_1'x_1' + v_2x_2'' + v_2'x_2'$$

Substituting x_p, x_p', and x_p'' into our differential equation, and recalling that x_1 and x_2 are solutions of the homogeneous equation, we get:

$$\begin{aligned}
x_p'' + px_p' + qx_p &= (v_1x_1'' + v_1'x_1' + v_2x_2'' + v_2'x_2') + p(v_1x_1' + v_2x_2') + q(v_1x_1 + v_2x_2) \\
&= v_1(x_1'' + px_1' + qx_1) + v_2(x_2'' + px_2' + qx_2) + v_1'x_1' + v_2'x_2' \\
&= v_1(0) + v_2(0) + v_1'x_1' + v_2'x_2' \\
&= v_1'x_1' + v_2'x_2' \\
&= f(t)
\end{aligned}$$

This gives us our second condition:

$$\boxed{v_1'x_1' + v_2'x_2' = f(t)}$$

Solving the System

We now have two equations for the derivatives of our variables v_1' and v_2':

$$\begin{cases} v_1'x_1 + v_2'x_2 = 0 \\ v_1'x_1' + v_2'x_2' = f(t) \end{cases}$$

While any method for solving this system works, Cramer's Rule provides the clearest approach:

$$v_1' = \frac{\begin{vmatrix} 0 & x_2 \\ f & x_2' \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}} = \frac{-x_2 f}{x_1x_2' - x_2x_1'} = \frac{-x_2 f}{W(x_1, x_2)}$$

$$v_2' = \frac{\begin{vmatrix} x_1 & 0 \\ x_1' & f \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}} = \frac{x_1 f}{x_1x_2' - x_2x_1'} = \frac{x_1 f}{W(x_1, x_2)}$$

Note that since $\{x_1, x_2\}$ is a fundamental solution set, the Wronskian $W(x_1, x_2)$ is nonzero. This gives us formulas we can integrate to find v_1 and v_2:

$$\boxed{v_1 = \int \left(\frac{-x_2 f}{W(x_1, x_2)}\right) dt} \quad \text{and} \quad \boxed{v_2 = \int \left(\frac{x_1 f}{W(x_1, x_2)}\right) dt}$$

Example 1

Let's solve the differential equation:

$$x'' + x = \tan t$$

First, we solve the homogeneous equation $x'' + x = 0$ to get the general solution $x(t) = C_1 \cos t + C_2 \sin t$. Let's set $x_1 = \cos t$ and $x_2 = \sin t$.

The Wronskian is:

$$\begin{aligned} W(\cos t, \sin t) &= \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} \\ &= (\cos t)(\cos t) - (\sin t)(-\sin t) \\ &= \cos^2 t + \sin^2 t = 1 \end{aligned}$$

Now we can apply our formulas:

$$\begin{aligned} v_1 &= \int \frac{-(\sin t)(\tan t)}{1} dt \\ &= \int \frac{-\sin^2 t}{\cos t} dt \\ &= \sin t - \ln|\sec t + \tan t| \end{aligned}$$

$$\begin{aligned} v_2 &= \int \frac{(\cos t)(\tan t)}{1} dt \\ &= \int \sin t \, dt \\ &= -\cos t \end{aligned}$$

Substituting back into our particular solution:

$$\begin{aligned} x_p &= v_1 x_1 + v_2 x_2 \\ &= (\sin t - \ln|\sec t + \tan t|)(\cos t) + (-\cos t)(\sin t) \\ &= \sin t \cos t - \cos t \ln|\sec t + \tan t| - \sin t \cos t \\ &= -\cos t \ln|\sec t + \tan t| \end{aligned}$$

Therefore, the general solution is:

$$x(t) = C_1 \cos t + C_2 \sin t - \cos t \ln|\sec t + \tan t|$$

Note: If we include the constants of integration in v_1 and v_2, we can derive the general solution directly without explicitly using the superposition principle.

Practice Examples

Solve the following differential equations using variation of parameters:

1. $x'' - 5x' + 6x = 2e^t$
2. $x'' + 2x' + x = 3e^{-t}$

Homework Assignment

Solve the differential equations using variation of parameters:

1. $x'' + 4x = \tan(2t)$
2. $x'' - 2x' + x = t^{-1}e^t$
3. $x'' + 16x = \sec(4t)$
4. $4x'' - 4x' + x = 16e^{t/2}$
5. $x'' + x = \sec^3 t$
6. $\frac{1}{2}x'' + 2x = \tan(2t) - \frac{1}{2}e^t$
7. $x'' + x = \sec t$
8. $x'' + 9x = \sec^2(3t)$
9. $x'' + 4x' + 4x = e^{-2t} \ln t$

Extra Credit Problems:

1. $x'' + x = 3\sec t - t^2 + 1$
2. $x''' - x'' + x' - x = e^{-t}$

Differential Equations I (Math 330)

Course Notes: The Cauchy-Euler Equation

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The Cauchy-Euler Equation

In this section, we examine differential equations of the form:

$$at^2x'' + btx' + cx = 0$$

We can rewrite this slightly as:

$$x'' + \left(\frac{b}{a}\right)\frac{1}{t}x' + \left(\frac{c}{at^2}\right)x = 0$$

This reframing helps us observe that we'll only consider solutions away from $t = 0$. Also, since $a \neq 0$, we can always divide by it.

Variable Transformation Technique

The technique we'll use—transforming our independent variable—is powerful and appears frequently in differential equations, statistics, and other mathematical disciplines.

Let's set:

$$s = \ln t$$

Then:

$$t = e^s \quad \text{and} \quad \frac{ds}{dt} = \frac{1}{t} = e^{-s}$$

If we define $Y(s) \equiv x(t)$, then we must use the chain rule to differentiate Y as a function of t:

$$x'(t) = \frac{d}{dt}[x(t)] = \frac{d}{dt}[Y(s(t))] = \frac{dY}{ds}\frac{ds}{dt} = Y'(s)e^{-s}$$

Continuing with this approach and applying the product rule:

$$\begin{aligned} x''(t) &= \frac{d}{dt}[x'(t)] \\ &= \frac{d}{dt}(Y'(s)e^{-s}) \\ &= \frac{d}{dt}(Y'(s)e^{-s})\frac{ds}{dt} \\ &= [Y''(s)e^{-s} + Y'(s)(-e^{-s})]e^{-s} \\ &= e^{-2s}[Y''(s) - Y'(s)] \end{aligned}$$

When we substitute these expressions for x, x', and x'' back into our original equation, we get:

$$\begin{aligned} at^2[e^{-2s}(Y''(s) - Y'(s))] + bt[Y'(s)e^{-s}] + c[Y(s)] &= 0 \\ a(e^{2s})[e^{-2s}(Y''(s) - Y'(s))] + b(e^s)[Y'(s)e^{-s}] + c[Y(s)] &= 0 \\ a[Y''(s) - Y'(s)] + b[Y'(s)] + c[Y(s)] &= 0 \end{aligned}$$

This simplifies to:

$$\boxed{aY''(s) + (b-a)Y'(s) + cY(s) = 0}$$

Now we have a constant-coefficient second-order differential equation with the characteristic polynomial:

$$Q(r) = ar^2 + (b-a)r + c$$

Determining the Solutions

The solutions of this equation give us $Y(s) = e^{rs}$, where r is a root of $Q(r)$. Recalling our transformation:

$$e^{rs} = e^{r\ln t} = t^r$$

We have similar results for other types of solutions (repeated roots and complex pairs).

Solution Summary

Discriminant	**Roots of $Q(r)$**	**General Solution**
$K = (b-a)^2 - 4ac > 0$	Two real roots, $r_1 \neq r_2$	$x(t) = C_1 t^{r_1} + C_2 t^{r_2}$
$K = (b-a)^2 - 4ac = 0$	Single real root, r (mult 2)	$x(t) = C_1 t^{r} + C_2 \ln(t) t^{r}$
$K = (b-a)^2 - 4ac < 0$	Complex pair, $r = \alpha \pm \beta i$	$x(t) = C_1 t^{\alpha} \cos(\beta \ln t) + C_2 t^{\alpha} \sin(\beta \ln t)$

Practice Examples

Solve the following differential equations for $t > 0$:

1. $t^2 x'' + 7tx' - 7x = 0$
2. $t^2 x'' + 5tx' + 4x = 0$
3. $t^2 x'' + 7tx' + 5x = 0$, $x(1) = -1$, $x'(1) = 13$

Homework Assignment

Solve the following differential equations using the Cauchy-Euler method for $t > 0$:

1. $t^2 x'' + 2tx' - 6x = 0$
2. $x'' + \frac{6}{t}x' + \frac{4}{t^2}x = 0$
3. $9t^2 x'' + 15tx' + x = 0$
4. $t^2 x'' - 3tx' + 4x = 0$
5. $t^2 x'' - 4tx' + 4x = 0$, $x(1) = -2$, $x'(1) = -11$

Differential Equations: Inverse Laplace Transforms & Partial Fractions

Shalmali Bandyopadhyay

Understanding Inverse Laplace Transforms

Having established our table of Laplace transforms, we now require a method to reverse the process—converting functions from s-domain back to the original t-domain. This reverse operation, known as the Inverse Laplace Transform, allows us to obtain explicit solutions to differential equations.

While the theoretical computation of inverse transforms involves complex contour integration, our practical approach will focus on manipulating expressions in the s-domain to match patterns from our Laplace transform table. We can then "read backward" from the table to determine the corresponding time-domain function.

To facilitate this process, we need to master partial fraction decomposition—an algebraic technique that breaks complicated rational expressions into simpler components that directly correspond to known inverse Laplace transforms.

Fundamentals of Partial Fraction Decomposition

Consider a proper rational function $X(s) = \frac{P(s)}{Q(s)}$ where the degree of $P(s)$ is less than the degree of $Q(s)$. Our goal is to express this function as a sum of simpler rational expressions.

The foundation of this technique relies on the fundamental theorem of algebra, which guarantees that any polynomial with real coefficients can be expressed as a product of:

- Linear factors of the form $(s - r)$ where r is a real root
- Irreducible quadratic factors of the form $(as^2 + bs + c)$ where $b^2 - 4ac < 0$

Thus, our denominator $Q(s)$ can be written as:

$$Q(s) = (s - r_1)^{m_1} \cdots (s - r_k)^{m_k} (a_1 s^2 + b_1 s + c_1)^{n_1} \cdots (a_l s^2 + b_l s + c_l)^{n_l}$$

Each factor in this representation requires a specific decomposition approach.

Three Essential Decomposition Patterns

Pattern 1: Non-repeated Linear Factors

When $Q(s)$ contains a non-repeated linear factor $(s - r)$, the partial fraction expansion includes a term of the form:

$$\frac{A}{s - r}$$

Here, A is a constant coefficient we must determine. This form corresponds directly to the inverse transform Ae^{rt}.

Pattern 2: Non-repeated Quadratic Factors

For each irreducible quadratic factor $(as^2 + bs + c)$ appearing in $Q(s)$, the decomposition contains a term of the form:

$$\frac{As + B}{as^2 + bs + c}$$

The numerator has this specific linear form to accommodate both sine and cosine terms that will appear in the inverse transform.

Pattern 3: Repeated Factors

When factors appear with multiplicities greater than one, additional terms are required:

- For a linear factor $(s - r)^m$ with multiplicity m, we include the series:

$$\frac{A_1}{s - r} + \frac{A_2}{(s - r)^2} + \cdots + \frac{A_m}{(s - r)^m}$$

- For a quadratic factor $(as^2 + bs + c)^n$ with multiplicity n, the expansion contains:

$$\frac{A_1 s + B_1}{as^2 + bs + c} + \frac{A_2 s + B_2}{(as^2 + bs + c)^2} + \cdots + \frac{A_n s + B_n}{(as^2 + bs + c)^n}$$

Solving for Coefficients

To determine the unknown coefficients in our decomposition, we combine all terms under a common denominator and compare the resulting numerator with the original $P(s)$. This comparison yields a system of linear equations that can be solved to find all constants.

The number of coefficients to determine equals the degree of the denominator polynomial $Q(s)$, ensuring a well-defined system.

Converting to Standard Form for Inverse Transforms

After obtaining the partial fraction decomposition, we often need additional algebraic manipulation to match forms in our Laplace transform table.

Working with Repeated Linear Factors

When dealing with terms like $\frac{A}{(s-r)^k}$, we need to recognize that the corresponding inverse transform takes the form:

$$\mathcal{L}^{-1}\left\{\frac{A}{(s - r)^k}\right\} = A\frac{t^{k-1}}{(k - 1)!}e^{rt}$$

To match our table's format, we rewrite:

$$\frac{A}{(s - r)^k} = \frac{A(k - 1)!}{(k - 1)!(s - r)^k} = \frac{C}{(s - r)^k}$$

where $C = \frac{A}{(k-1)!}$, yielding the inverse transform $Ct^{k-1}e^{rt}$.

Working with Quadratic Factors

For terms involving irreducible quadratic factors, we complete the square in the denominator to obtain the standard form:

$$\frac{As+B}{as^2+bs+c} = \frac{C(s-\alpha)+D\beta}{(s-\alpha)^2+\beta^2}$$

where $\alpha = -\frac{b}{2a}$ and $\beta = \sqrt{\frac{4ac-b^2}{4a^2}}$.
The corresponding inverse transform becomes:

$$\mathcal{L}^{-1}\left\{\frac{C(s-\alpha)+D\beta}{(s-\alpha)^2+\beta^2}\right\} = e^{\alpha t}(C\cos(\beta t)+D\sin(\beta t))$$

Illustrative Examples

Determine the inverse Laplace transform of each expression using partial fraction decomposition:

1. $\frac{2s}{s^2-2s+5}$
2. $\frac{s^4-s^3+2s^2+2}{s^3(s^2+1)}$
3. $\frac{5s^3-39s^2+14s-32}{(s^2+1)(s^2-7s+10)}$
4. $\frac{s^2+3}{s^3-s^2+s-1}$

Homework Assignment

For each of the following rational functions, apply partial fraction decomposition to find the corresponding inverse Laplace transform:

1. $\frac{-s}{s^2+6s+9}$
2. $\frac{-2s+7}{s^2-s-2}$
3. $\frac{-s^2+2s+11}{(s-1)(s^2+6s+5)}$
4. $\frac{2s^2-7s+7}{(s-3)(s^2-4s+5)}$
5. $\frac{3s^3+10s^2-4s+8}{s^3(s^2+4)}$
6. $\frac{s^3+8s^2+15s-12}{s(s^3+4s^2+s-6)}$

Differential Equations: Solving IVPs using Laplace Transforms

Shalmali Bandyopadhyay

Solving IVPs using Laplace Transforms

Now that we have a Laplace transform table and the ability to "undo" the Laplace transform, we can now solve differential equations using this method.

1. Apply the Laplace transform to our given IVP.
2. Solve the new equation for $F(s)$.
3. Find the inverse Laplace transform via table look up and partial fractions.

Examples

Solve the given IVP using the Laplace transform method.

Example 1: $x'' - 2x' + 5x = 0, \quad x(0) = 2, \quad x'(0) = 4$

Proof. First, apply the Laplace transform to the differential equation:

$$\mathcal{L}\{x'' - 2x' + 5x\} = \mathcal{L}\{0\}$$

Using the Laplace transform properties:

$$s^2X(s) - sx(0) - x'(0) - 2[sX(s) - x(0)] + 5X(s) = 0$$

Substitute the initial conditions $x(0) = 2$ and $x'(0) = 4$:

$$s^2X(s) - 2s - 4 - 2[sX(s) - 2] + 5X(s) = 0$$

Simplify:

$$\begin{aligned} s^2X(s) - 2s - 4 - 2sX(s) + 4 + 5X(s) &= 0 \\ s^2X(s) - 2sX(s) + 5X(s) &= 2s \\ X(s)(s^2 - 2s + 5) &= 2s \\ X(s) &= \frac{2s}{s^2 - 2s + 5} \end{aligned}$$

To find the inverse Laplace transform, we complete the square in the denominator:

$$s^2 - 2s + 5 = (s-1)^2 + 4$$

So we have:

$$X(s) = \frac{2s}{(s-1)^2 + 4} = \frac{2s}{(s-1)^2 + 2^2}$$

We can rewrite this as:

$$\begin{aligned} X(s) &= \frac{2[(s-1)+1]}{(s-1)^2 + 2^2} \\ &= \frac{2(s-1)}{(s-1)^2 + 2^2} + \frac{2}{(s-1)^2 + 2^2} \end{aligned}$$

Using Laplace transform pairs:

$$\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos(bt)$$
$$\mathcal{L}^{-1}\left\{\frac{b}{(s-a)^2+b^2}\right\} = e^{at}\sin(bt)$$

With $a = 1$ and $b = 2$, we get:

$$x(t) = 2e^t\cos(2t) + e^t\sin(2t)$$

□

Example 2: $x'' + 6x' + 9x = 0, \quad x(0) = -1, \quad x'(0) = 6$

Proof. Apply the Laplace transform:

$$\mathcal{L}\{x'' + 6x' + 9x\} = \mathcal{L}\{0\}$$

Using the Laplace transform properties:

$$s^2X(s) - sx(0) - x'(0) + 6[sX(s) - x(0)] + 9X(s) = 0$$

Substitute the initial conditions $x(0) = -1$ and $x'(0) = 6$:

$$s^2X(s) + s - 6 + 6[sX(s) + 1] + 9X(s) = 0$$

Simplify:

$$\begin{aligned} s^2X(s) + s - 6 + 6sX(s) + 6 + 9X(s) &= 0 \\ s^2X(s) + 6sX(s) + 9X(s) &= -s \\ X(s)(s^2 + 6s + 9) &= -s \\ X(s) &= \frac{-s}{(s+3)^2} \end{aligned}$$

We can rewrite this as:

$$\begin{aligned} X(s) &= \frac{-s}{(s+3)^2} \\ &= -\frac{s+3-3}{(s+3)^2} \\ &= -\frac{s+3}{(s+3)^2} + \frac{3}{(s+3)^2} \end{aligned}$$

Using Laplace transform pairs:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2}\right\} = te^{at}$$
$$\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2}\right\} = e^{at}$$

With $a = -3$, we get:

$$\begin{aligned} x(t) &= -e^{-3t} + 3te^{-3t} \\ &= e^{-3t}(3t - 1) \end{aligned}$$

□

Example 3: $x'' + x = t^2 + 2, \quad x(0) = 1, \quad x'(0) = -1$

Proof. Apply the Laplace transform:

$$\mathcal{L}\{x'' + x\} = \mathcal{L}\{t^2 + 2\}$$

Using the Laplace transform properties:

$$s^2 X(s) - sx(0) - x'(0) + X(s) = \frac{2}{s^3} + \frac{2}{s}$$

Substitute the initial conditions $x(0) = 1$ and $x'(0) = -1$:

$$s^2 X(s) - s - (-1) + X(s) = \frac{2}{s^3} + \frac{2}{s}$$

Simplify:

$$\begin{aligned} s^2 X(s) - s + 1 + X(s) &= \frac{2}{s^3} + \frac{2}{s} \\ X(s)(s^2 + 1) &= \frac{2}{s^3} + \frac{2}{s} + s - 1 \end{aligned}$$

We need to find a common denominator on the right side:

$$\begin{aligned} X(s)(s^2 + 1) &= \frac{2}{s^3} + \frac{2}{s} + s - 1 \\ &= \frac{2}{s^3} + \frac{2s^2}{s^3} + \frac{s^4}{s^3} - \frac{s^3}{s^3} \\ &= \frac{2 + 2s^2 + s^4 - s^3}{s^3} \end{aligned}$$

Solving for $X(s)$:

$$X(s) = \frac{2 + 2s^2 + s^4 - s^3}{s^3(s^2 + 1)}$$

Let's decompose this into partial fractions. After solving for the coefficients, we get:

$$X(s) = \frac{1}{s} + \frac{1}{s^3} + \frac{s - 1}{s^2 + 1}$$

Using Laplace transform pairs:

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} &= 1 \\ \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} &= \frac{t^2}{2} \\ \mathcal{L}^{-1}\left\{\frac{s}{s^2 + 1}\right\} &= \cos(t) \\ \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} &= \sin(t) \end{aligned}$$

Combining these results:

$$x(t) = 1 + \frac{t^2}{2} + \cos(t) - \sin(t)$$

□

Homework Assignment

Solve the given IVP using the Laplace transform method.

1. $x'' - x' - 2x = 0, \quad x(0) = -2, \quad x'(0) = 5$
2. $x'' + 6x' + 5x = 12e^t, \quad x(0) = -1, \quad x'(0) = 7$
3. $x'' - 4x' + 5x = 4e^{3t}, \quad x(0) = 2, \quad x'(0) = 7$
4. $x'' + 4x = 4t^2 - 4t + 10, \quad x(0) = 0, \quad x'(0) = 3$
5. $x''' + 4x'' + x' - 6x = -12, \quad x(0) = 1, \quad x'(0) = 4, \quad x''(0) = -2$
6. $x''' + x'' + 3x' - 5x = 16e^{-t}, \quad x(0) = 0, \quad x'(0) = 2, \quad x''(0) = -4$
7. $x'' + 9x = 6\sin(3t), \quad x(0) = 0, \quad x'(0) = 0$
8. $x'' + 4x' + 4x = t^3e^{-2t}, \quad x(0) = 2, \quad x'(0) = -3$
9. $x'' + 2x' + 5x = te^{-t}\cos(2t), \quad x(0) = 0, \quad x'(0) = 0$
10. $x''' - 3x'' + 3x' - x = t^2, \quad x(0) = 0, \quad x'(0) = 0, \quad x''(0) = 0$

Differential Equations: Numerical Methods

Shalmali Bandyopadhyay

Introduction to Numerical Approximations

When analytical solutions to initial-value problems (IVPs) are difficult or impossible to find, we turn to numerical methods. These techniques provide approximate solutions at discrete points for problems of the form:

$$y'(t) = f(t, y), \quad y(t_0) = y_0 \tag{1}$$

A numerical method generates a sequence of points:

$$\{(t_0, y_0), (t_1, y_1), \ldots, (t_N, y_N)\} \tag{2}$$

where $t_j = t_0 + j\Delta t$ and y_j approximates the true solution value at t_j.

Euler's Method

Euler's Method is the simplest numerical approach, using a linear approximation based on the derivative at each step.

Algorithm for Euler's Method

Given $y' = f(t, y)$, $y(t_0) = y_0$, to find approximate values on $[t_0, t_{max}]$:

1. Choose step size $\Delta t = \frac{t_{max} - t_0}{N}$ with N a positive integer
2. For $j = 0, 1, \ldots, N-1$ compute:
   - $y_{j+1} = y_j + f(t_j, y_j)\Delta t$
   - $t_{j+1} = t_j + \Delta t$
3. The points (t_j, y_j) form a piecewise linear approximation to $y(t)$

Example: Euler's Method

Let's approximate the solution to $y' = t + y$, $y(0) = 2$ on $[0, 1]$ using $\Delta t = 0.25$ (so $N = 4$).

j	t_j	y_j	$f(t_j, y_j) = t_j + y_j$	$y_{j+1} = y_j + f(t_j, y_j)\Delta t$
0	0	2	$0 + 2 = 2$	$y_1 = 2 + 2 \cdot 0.25 = 2.5$
1	0.25	2.5	$0.25 + 2.5 = 2.75$	$y_2 = 2.5 + 2.75 \cdot 0.25 = 3.1875$
2	0.5	3.1875	$0.5 + 3.1875 = 3.6875$	$y_3 = 3.1875 + 3.6875 \cdot 0.25 = 4.109375$
3	0.75	4.109375	$0.75 + 4.109375 = 4.859375$	$y_4 = 4.109375 + 4.859375 \cdot 0.25 = 5.32421875$
4	1.0	5.32421875		

Improved Euler's Method (Heun's Method)

Heun's Method improves accuracy by averaging the slopes at the current point and the point estimated by Euler's Method.

Algorithm for Improved Euler's Method

Given $y' = f(t, y)$, $y(t_0) = y_0$, to find approximate values on $[t_0, t_{max}]$:

1. Choose step size $\Delta t = \frac{t_{max} - t_0}{N}$ with N a positive integer

2. For $j = 0, 1, \ldots, N-1$ compute:

   - $k_1 = f(t_j, y_j)$
   - $\tilde{y}_{j+1} = y_j + k_1 \Delta t$
   - $t_{j+1} = t_j + \Delta t$
   - $k_2 = f(t_{j+1}, \tilde{y}_{j+1})$
   - $k_{avg} = \frac{k_1 + k_2}{2}$
   - $y_{j+1} = y_j + k_{avg} \Delta t$

Example: Improved Euler's Method

Let's approximate the solution to $y' = t + y$, $y(0) = 2$ on $[0, 1]$ using $\Delta t = 0.25$ (so $N = 4$).

j	t_j	y_j	k_1	$\tilde{y}_{j+1}$	k_2	k_{avg}	y_{j+1}
0	0	2	2	2.5	2.75	2.375	2.59375
1	0.25	2.59375	2.84375	3.3046875	3.5546875	3.1992188	3.3935547
2	0.5	3.3935547	3.8935547	4.3669434	5.1169434	4.5052491	4.5199170
3	0.75	4.5199170	5.2699170	5.8373962	6.5873962	5.9286566	5.9020811
4	1.0	5.9020811					

Fourth-Order Runge-Kutta Method

The Runge-Kutta Method offers higher accuracy by evaluating the function at intermediate points.

Algorithm for Fourth-Order Runge-Kutta Method

Given $y' = f(t, y)$, $y(t_0) = y_0$, to find approximate values on $[t_0, t_{max}]$:

1. Choose step size $\Delta t = \frac{t_{max} - t_0}{N}$ with N a positive integer

2. For $j = 0, 1, \ldots, N-1$ compute:

   - $k_1 = f(t_j, y_j)$
   - $k_2 = f(t_j + \frac{\Delta t}{2}, y_j + \frac{k_1 \Delta t}{2})$
   - $k_3 = f(t_j + \frac{\Delta t}{2}, y_j + \frac{k_2 \Delta t}{2})$
   - $k_4 = f(t_j + \Delta t, y_j + k_3 \Delta t)$
   - $y_{j+1} = y_j + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
   - $t_{j+1} = t_j + \Delta t$

Example: Fourth-Order Runge-Kutta Method

Let's approximate the solution to $y' = t + y$, $y(0) = 2$ on $[0, 1]$ using $\Delta t = 0.25$ (so $N = 4$).

j	t_j	y_j	k_1	k_2	k_3	k_4	y_{j+1}
0	0	2	2	2.375	2.4375	2.9375	2.6041667
1	0.25	2.6041667	2.8541667	3.2864583	3.3489583	3.8489583	3.4192708
2	0.5	3.4192708	3.9192708	4.4192708	4.4817708	5.0442708	4.4625651
3	0.75	4.4625651	5.2125651	5.7906901	5.8532476	6.4844976	5.8093590
4	1.0	5.8093590					

Comparison of Methods

For the differential equation $y' = t + y$, $y(0) = 2$, we can find the analytical solution:

$$y(t) = 2e^t - t - 1 \tag{3}$$

At $t = 1$, the exact solution is $y(1) = 2e^1 - 1 - 1 = 2e - 2 \approx 5.43656366$

Method	Approximation at $t = 1$	Absolute Error
Euler	5.32421875	0.11234491
Improved Euler	5.9020811	0.46551744
Runge-Kutta	5.8093590	0.37279534

Error Analysis in Numerical Methods

The accuracy of numerical methods can be characterized by their global truncation error:

- Euler's Method: Error $= O(\Delta t)$
- Improved Euler's Method: Error $= O(\Delta t^2)$
- Fourth-order Runge-Kutta: Error $= O(\Delta t^4)$

This means that if we halve the step size:

- Euler's error is approximately halved
- Improved Euler's error is approximately quartered
- Runge-Kutta's error is reduced by a factor of 16

Homework Exercises

1. For the IVP $\frac{dy}{dt} = y(3 - y)$, $y(0) = 2$:
   (a) Find the analytic solution
   (b) Use Euler's Method on $[0, 2]$ with $\Delta t = 0.25$
   (c) Use Improved Euler's Method on $[0, 2]$ with $\Delta t = 0.4$
   (d) Use the fourth-order Runge-Kutta Method on $[0, 2]$ with $\Delta t = 0.5$
   (e) Compare all results graphically
2. For the IVP $\frac{dy}{dt} = 1 + t\cos(t^2)$, $y(0) = 1$:
   (a) Use Euler's Method on $[0, 0.5]$ with $N = 5$
   (b) Use Improved Euler's Method on $[0, 1]$ with $N = 4$
   (c) Use the fourth-order Runge-Kutta Method on $[0, \frac{3}{4}]$ with $N = 3$
3. Investigate the stability of Euler's Method applied to $y' = -10y$ using different step sizes. For which values of Δt does the solution remain stable?
4. For the system of differential equations:

$$\frac{dx}{dt} = y \tag{4}$$

$$\frac{dy}{dt} = -\sin(x) \tag{5}$$

   with initial conditions $x(0) = 0$, $y(0) = 1$:

(a) Use the fourth-order Runge-Kutta Method on $[0, 10]$ with $\Delta t = 0.1$

(b) Interpret the physical meaning of this system (simple pendulum)

Differential Equations: Numerical Methods

Shalmali Bandyopadhyay

Series Solutions for Differential Equations

In this section, we examine approaches for solving differential equations of the form:

$$x'' + p(t)x' + q(t)x = f(t) \tag{1}$$

Previous methods revealed that solutions often manifest as exponential, polynomial, or sinusoidal functions, possibly in combination. Each of these solution types can be represented using Taylor series, which gives us insight into the power of this technique. Historically, series methods were predominant before computational advancements.

Analytic Functions

A function $p(t)$ is considered **analytic** at $t = t_0$ if it can be expanded as a Taylor series:

$$p(t) = \sum_{n=0}^{\infty} \frac{p^{(n)}(t_0)}{n!}(t - t_0)^n = p(t_0) + \frac{p'(t_0)(t - t_0)}{1!} + \frac{p''(t_0)(t - t_0)^2}{2!} + \cdots \tag{2}$$

that converges within some interval around $t = t_0$.

When addressing differential equations using series solutions, we assume both $p(t)$ and $q(t)$ are analytic functions for all values of t. We then propose a series solution for $x(t)$. For simplicity, we typically employ a Maclaurin series (Taylor series centered at $t_0 = 0$):

$$x(t) = \sum_{n=0}^{\infty} a_n t^n, \quad a_n = \frac{x^{(n)}(0)}{n!} \tag{3}$$

The standard approach involves substituting this proposed solution and its derivatives into the original differential equation, then determining the formula for coefficients a_n to obtain the analytic solution.

Example: Series Solution Method

Problem: Solve the differential equation using series solution around $t = t_0$.

$$2x'' + tx' + x = 0 \tag{4}$$

Solution: We begin by expressing $x(t)$ as a power series:

$$x(t) = \sum_{n=0}^{\infty} a_n t^n \tag{5}$$

We then compute the derivatives:

$$x'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}, \quad x''(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} \tag{6}$$

Substituting into the original equation:

$$2x'' + tx' + x = 2\left(\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}\right) + t\left(\sum_{n=1}^{\infty} na_n t^{n-1}\right) + \sum_{n=0}^{\infty} a_n t^n \quad (7)$$

$$= 2\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=1}^{\infty} na_n t^n + \sum_{n=0}^{\infty} a_n t^n \quad (8)$$

To facilitate addition, we adjust the indices so all terms have t^n:

$$= \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}t^n + \sum_{n=1}^{\infty} na_n t^n + \sum_{n=0}^{\infty} a_n t^n \quad (9)$$

Since the summations have different starting indices, we separate the $n = 0$ terms from the first and third sums:

$$= 2(0+2)(0+1)a_{0+2}t^0 + \sum_{n=1}^{\infty} 2(n+2)(n+1)a_{n+2}t^n + \sum_{n=1}^{\infty} na_n t^n + a_0 t^0 + \sum_{n=1}^{\infty} a_n t^n \quad (10)$$

$$= (4a_2 + a_0) + \sum_{n=1}^{\infty}[2(n+2)(n+1)a_{n+2} + na_n + a_n]t^n \quad (11)$$

$$= (4a_2 + a_0) + \sum_{n=1}^{\infty}[2(n+2)(n+1)a_{n+2} + (n+1)a_n]t^n \quad (12)$$

Since the right side of the original equation equals zero, and a power series equals zero only when all coefficients are zero, we obtain:

$$\begin{cases} 4a_2 + a_0 = 0 \\ 2(n+2)(n+1)a_{n+2} + (n+1)a_n = 0 \end{cases} \quad (13)$$

For an initial value problem with given $x(0) = a_0$ and $x'(0) = a_1$, we can rewrite this as:

$$\begin{cases} a_2 = -\frac{a_0}{4} \\ a_{n+2} = -\frac{1}{2(n+2)}a_n, \quad n \geq 1 \end{cases} \quad (14)$$

To determine the pattern for coefficients:

$$a_0 = \text{initial condition} \quad (15)$$

$$a_1 = \text{initial condition} \quad (16)$$

$$a_2 = -\frac{a_0}{4} \quad (17)$$

$$a_3 = -\frac{a_1}{6} \quad (18)$$

$$a_4 = -\frac{a_2}{8} = -\frac{1}{8}\left(-\frac{a_0}{4}\right) = \frac{a_0}{32} \quad (19)$$

$$a_5 = -\frac{a_3}{10} = -\frac{1}{10}\left(-\frac{a_1}{6}\right) = \frac{a_1}{60} \quad (20)$$

$$a_6 = -\frac{a_4}{12} = -\frac{1}{12}\left(\frac{a_0}{32}\right) = -\frac{a_0}{384} \quad (21)$$

$$a_7 = -\frac{a_5}{14} = -\frac{1}{14}\left(\frac{a_1}{60}\right) = -\frac{a_1}{840} \quad (22)$$

$$a_8 = -\frac{a_6}{16} = -\frac{1}{16}\left(-\frac{a_0}{384}\right) = \frac{a_0}{6144} \quad (23)$$

Examining these coefficients, we identify patterns for even and odd indices:
For even-indexed terms, $n \geq 1$:

$$a_{2n} = \frac{(-1)^n a_0}{4 \cdot 8 \cdot 12 \cdots (4n)} = \frac{(-1)^n a_0}{4^n \cdot n!} \tag{24}$$

For odd-indexed terms, $n \geq 1$:

$$a_{2n+1} = \frac{(-1)^n a_1}{6 \cdot 10 \cdot 14 \cdots (4n+2)} = \frac{(-1)^n a_1}{2 \cdot 6 \cdot 10 \cdot 14 \cdots (4n+2)} \tag{25}$$

This gives us two linearly independent solutions:

$$x_1(t) = 1 - \frac{t^2}{4} + \frac{t^4}{32} - \frac{t^6}{384} + \frac{t^8}{6144} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{4^n \cdot n!} \tag{26}$$

$$x_2(t) = t - \frac{t^3}{6} + \frac{t^5}{60} - \frac{t^7}{840} + \cdots \tag{27}$$

Thus, our general solution is $x(t) = a_0 x_1(t) + a_1 x_2(t)$.

Additional Examples

Problem 1: Solve using series solution around $t = 0$.

$$x'' - tx' + 4x = 0 \tag{28}$$

Problem 2: Solve using series solution around $t = 0$.

$$x'' - t = 0 \tag{29}$$

Problem 3: Solve using series solution around $t = 0$.

$$(t^2 + 1)x'' - tx' + x = 0 \tag{30}$$

Homework Set A

Solve the following differential equations using series solutions around $t = 0$.
Problem 1: $x'' + x = 0$
Problem 2: $x'' - t^2 x' - t = 0$

Approximation Methods for Higher-Order Equations

We now explore two different approaches for approximating solutions to second-order and higher-order differential equations: the Series Approximation Method and Higher-order Euler's Method.

Series Approximation Method

Recalling the general form of a Taylor series:

$$x(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n, \quad a_n = \frac{x^{(n)}(t_0)}{n!} \tag{31}$$

We use this representation to create an analytic approximation to an initial value problem (IVP) that converges within some interval around $t = t_0$.

As with other approximation techniques, the accuracy diminishes as we move away from the initial point. While we could explore error estimation in depth, we'll focus on the practical application.

Example: Series Approximation Technique

Problem: Find a sixth-order series approximation to the IVP:

$$x'' + tx' + x = 0, \quad x(0) = 1, \quad x'(0) = 0 \tag{32}$$

Solution: The initial conditions provide the first two terms of our Taylor series. We can determine the third term using the differential equation:

$$x'' = -x - tx' \implies x''(0) = -x(0) - 0 \cdot x'(0) = -1 \tag{33}$$

We establish a table for our coefficient calculations:

n	$x^{(n)}(0)$	$a_n = \frac{x^{(n)}(0)}{n!}$	Term in expansion, $a_n t^n$
0	1	$\frac{1}{0!} = 1$	1
1	0	$\frac{0}{1!} = 0$	0
2	-1	$\frac{-1}{2!} = -\frac{1}{2}$	$-\frac{1}{2}t^2$

To find higher-order derivatives, we differentiate x'' successively:

$$x''' = \frac{d}{dt}(x'') = \frac{d}{dt}(-x - tx') = -x' - x' - tx'' = -2x' - tx'' \tag{34}$$

$$x^{(4)} = -2x'' - tx''' - x'' = -3x'' - tx''' \tag{35}$$

$$x^{(5)} = -3x''' - tx^{(4)} - x''' = -4x''' - tx^{(4)} \tag{36}$$

$$x^{(6)} = -4x^{(4)} - tx^{(5)} - x^{(4)} = -5x^{(4)} - tx^{(5)} \tag{37}$$

Continuing our table:

n	$x^{(n)}(0)$	$a_n = \frac{x^{(n)}(0)}{n!}$	Term in expansion, $a_n t^n$
3	$-2x'(0) - 0 \cdot x''(0) = 0$	$\frac{0}{3!} = 0$	0
4	$-3x''(0) - 0 \cdot x'''(0) = 3$	$\frac{3}{4!} = \frac{1}{8}$	$\frac{1}{8}t^4$
5	$-4x'''(0) - 0 \cdot x^{(4)}(0) = 0$	$\frac{0}{5!} = 0$	0
6	$-5x^{(4)}(0) - 0 \cdot x^{(5)}(0) = -15$	$\frac{-15}{6!} = -\frac{1}{48}$	$-\frac{1}{48}t^6$

Therefore, our sixth-order approximation is:

$$x(t) \approx T_6(t) = 1 - \frac{1}{2}t^2 + \frac{1}{8}t^4 - \frac{1}{48}t^6 \tag{38}$$

This function represents an approximation to the solution of the differential equation, satisfying the given initial conditions.

Note that including more terms enhances the approximation's accuracy. The Runge-Kutta method discussed earlier produces a numerical approximation equivalent to a fourth-order series approximation (accurate to the first five terms of the Taylor series).

Higher-order Euler's Method

To apply Euler's method to second-order equations, we must first convert them into first-order systems. Here's how:

Example: Convert the second-order IVP into a system of first-order IVPs:

$$x'' + tx' + x = 0, \quad x(0) = 1, \quad x'(0) = 0 \tag{39}$$

Solution: Let $y = x'$. Then:

$$y' = x'' = -tx' - x = -ty - x \tag{40}$$

This gives us the system:

$$\begin{cases} x' = y \\ y' = -ty - x \end{cases} \tag{41}$$

with initial conditions $x(0) = 1$, $y(0) = 0$.

Euler's Algorithm for Systems

Given a system $x' = y$, $y' = F(t, x, y)$ with initial conditions $x(t_0) = x_0$, $y(t_0) = x'(t_0) = y_0$, we use this algorithm to find approximate values of $x(t)$ and $y(t)$ on $[t_0, t_{max}]$:

1. Choose step size $\Delta t = \frac{t_{max} - t_0}{N}$, where N is a positive integer.
2. For $j = 0, 1, \ldots, N-1$, compute:

$$x_{j+1} = x_j + y_j \Delta t \tag{42}$$

$$y_{j+1} = y_j + F(t_j, x_j, y_j)\Delta t \tag{43}$$

$$t_{j+1} = t_j + \Delta t \tag{44}$$

3. Plot the points (t_j, x_j) for $j = 0, 1, \ldots, N$ to create a piecewise linear approximation.

Example: Applying Euler's Method for Systems

Continuing with our example, we approximate the solution on $[0, 1]$ with $N = 5$ steps, giving $\Delta t = 0.2$.

j	t_j	x_j	y_j	$x_{j+1} = x_j + y_j \Delta t$	$y_{j+1} = y_j + F(t_j, x_j, y_j)\Delta t$
0	0	1	0	$1 + 0 \cdot 0.2 = 1$	$0 + (-0 \cdot 0 - 1) \cdot 0.2 = -0.2$
1	0.2	1	-0.2	$1 + (-0.2) \cdot 0.2 = 0.96$	$-0.2 + (-0.2 \cdot (-0.2) - 1) \cdot 0.2 = -0.392$
2	0.4	0.96	-0.392	$0.96 + (-0.392) \cdot 0.2 = 0.8816$	$-0.392 + (-0.4 \cdot (-0.392) - 0.96) \cdot 0.2 = -0.5666$
3	0.6	0.8816	-0.5666	$0.8816 + (-0.5666) \cdot 0.2 = 0.7683$	$-0.5666 + (-0.6 \cdot (-0.5666) - 0.8816) \cdot 0.2 = -0.7127$
4	0.8	0.7683	-0.7127	$0.7683 + (-0.7127) \cdot 0.2 = 0.6258$	$-0.7127 + (-0.8 \cdot (-0.7127) - 0.7683) \cdot 0.2 = -0.8208$
5	1.0	0.6258	-0.8208	–	–

Comparing Approximation Methods

Comparing the series approximation and Euler's method reveals interesting insights into the behavior of solutions. For visualization, we can plot both the curve from the series approximation and the points calculated via Euler's method.

For our previous example, the series approximation $T_6(t) = 1 - \frac{1}{2}t^2 + \frac{1}{8}t^4 - \frac{1}{48}t^6$ can be plotted alongside the points from Euler's method to demonstrate the relative accuracy of each approach.

Homework Set B

Find series approximations for the following IVPs. Use the first four nonzero terms in the expansion.

Problem 1: $x' = t - x^2, \quad x(0) = 1$

Problem 2: $x'' - t^3 x' + t x^2 = 0, \quad x(0) = -1, \quad x'(0) = 1$

Problem 3: $(2 + t^2)x'' - t x' + 4x = 0, \quad x(0) = -1, \quad x'(0) = 3$

Use Euler's Method for Systems to approximate these IVPs on $[0, 1]$**.**

Problem 4: (Problem 1 above)

Problem 5: (Problem 2 above)

Problem 6: (Problem 3 above)

Create comparative graphs showing both the series approximation curve and Euler's method points for each problem above.

Problem 7: (Problems 1 and 4 above)

Problem 8: (Problems 2 and 5 above)

Problem 9: (Problems 3 and 6 above)

Differential Equations: Existence and Uniqueness Theorems

Shalmali Bandyopadhyay

Existence and Uniqueness Theorems

Existence and Uniqueness Theorem

Given the differential equation $y' = f(t, y)$, if f is defined and continuous everywhere inside a rectangle $R = \{(t, y)|a \leq t \leq b, c \leq y \leq d\}$ containing the point (t_0, y_0), and if $\frac{\partial f}{\partial y}$ is continuous in R, then there exists a unique solution $y(t)$ passing through the point (t_0, y_0).

Lipschitz Condition

A function $f(t, y)$ satisfies a Lipschitz condition if for any two points (t, y_1) and (t, y_2), there exists a positive constant M such that:

$$|f(t, y_1) - f(t, y_2)| \leq M|y_2 - y_1| \tag{1}$$

Logistic Equation Example

Consider the differential equation $y' = y(1 - y)$ which models population growth with a carrying capacity.

The general solution is:

$$y(t) = \frac{1}{1 + Ce^{-t}} \tag{2}$$

The constant C is determined by the initial condition. The equation has two equilibrium solutions: $y(t) \equiv 0$ and $y(t) \equiv 1$.

- For $0 < y(0) < 1$: Solutions approach $y = 1$ as $t \to \infty$ (logistic growth)
- For $y(0) > 1$: Solutions decrease and approach $y = 1$ as $t \to \infty$
- For $y(0) < 0$: Solutions have a vertical asymptote and approach $y = 0$ as $t \to -\infty$

Additional Practice Problems

Approximate the solutions to the following differential equations using various techniques:

1. $\frac{dy}{dt} = \frac{y^2}{t}$, $y(1) = 2$, on the interval $[1, 2]$ with $\Delta t = 0.2$
2. $\frac{dy}{dt} = \sin(ty)$, $y(0) = 1$, on the interval $[0, 1]$ with $\Delta t = 0.25$