Numerical Implementation of the Sturm–Liouville Problem via Prüfer-Based Shooting

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The Discrete Sturm–Liouville Problem

Discrete Sturm–Liouville problem with Dirichlet boundary condition:

$$\begin{cases} \Delta_h(r_k \Delta_h x_k) + [\lambda w_k - q_k]\, x_{k+1} = 0, & k = 1, \dots, n-1 \\ x_0 = x_{n+1} = 0 \end{cases}$$

where

- $\Delta_h x_k = \dfrac{x_{k+1} - x_k}{t_{k+1} - t_k} = \dfrac{x_{k+1} - x_k}{h_k}$ is the **non-uniform step size**
- $r_k, w_k, q_k : \mathbb{R} \to \mathbb{R}$ are the **weight functions**, with $w_k > 0$, $r_k \neq 0$
- $\lambda \in \mathbb{R}$ is the eigenvalue parameter
- $r_k \Delta_h x_k$ is the quasiderivative

Test case. Set $r_k \equiv 1$, $w_k \equiv 1$, $q_k \equiv 0$:

$$\Delta_h(\Delta_h x_k) + \lambda\, x_{k+1} = 0, \qquad x_0 = x_{n+1} = 0.$$

Continuous analog $x'' + \lambda x = 0$ on $[0, \pi]$ has known eigenvalues $\lambda_n^c = n^2$, which we use as a ground truth.

Goal

Compare **regular shooting** and **Prüfer-based shooting** for computing eigenvalues on:

- a **uniform** grid,
- a **clustered** non-uniform grid, and
- a **graded** (geometric) non-uniform grid.

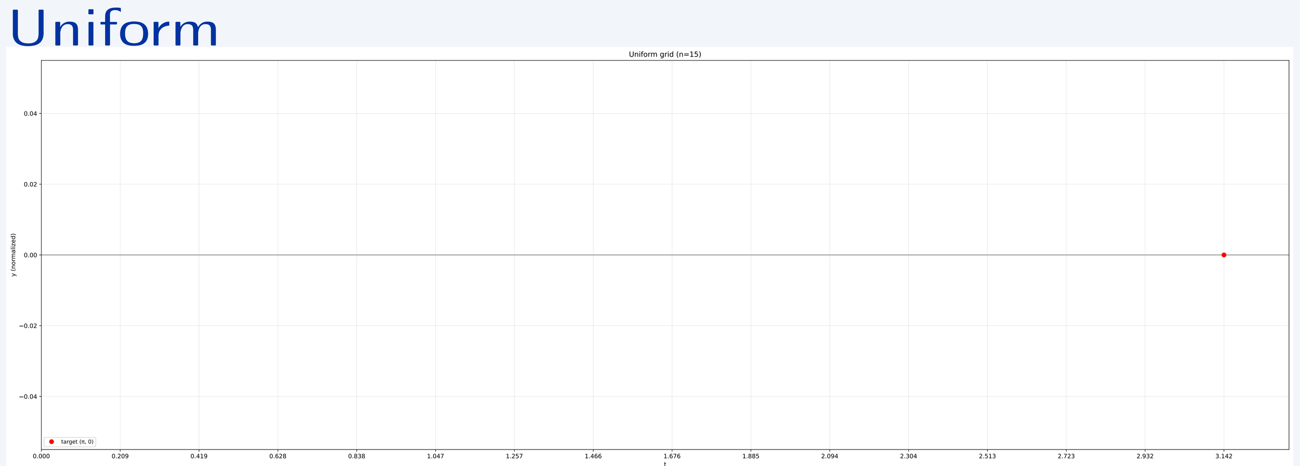
Accuracy is measured by how close the numerical eigenvalue λ_k is to $\lambda_k^c = k^2$.
Eigenfunctions are indexed by their zero count: ϕ_n has $n-1$ interior zeros.

Regular VS Prüfer

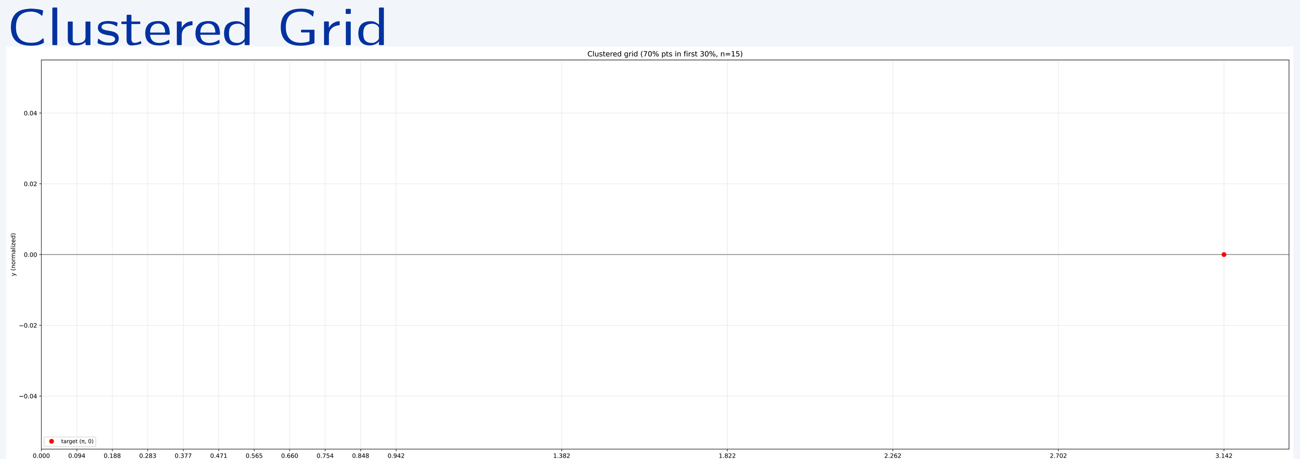
Accuracy $\equiv$ closeness of λ_n to $\lambda_n^c = n^2$

Visuals of Grids

Uniform



Clustered Grid



Graded Grid

Regular Shooting: From BVP to IVP

The obstacle is that $x_0 = 0$ at a and $x_{n+1} = 0$ at b are **two** conditions at different points.
The fix:

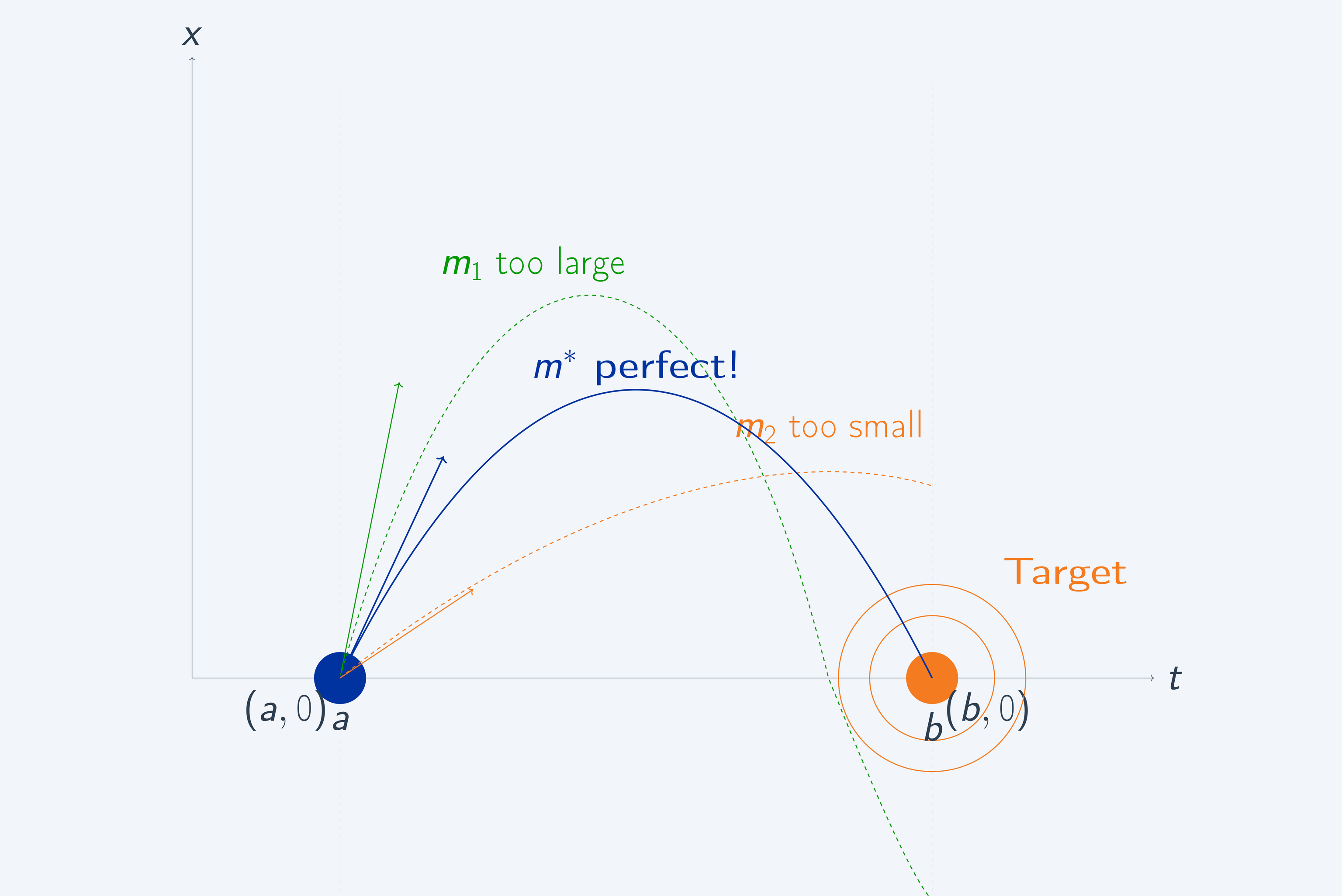
- Fix $x_0 = 0$, guess the initial discrete derivative $\Delta_h x_0 = x_1 / h_0$ (choose x_1).
- March forward via the recurrence

$$x_{k+2} = \left(1 + \frac{h_{k+1} r_k}{h_k r_{k+1}} - \frac{h_{k+1} h_k}{r_{k+1}} [\lambda w_k - q_k]\right) x_{k+1} - \frac{h_{k+1} r_k}{h_k r_{k+1}} x_k.$$

- Tune x_1 (equivalently the slope) until $x_{n+1} = 0$.

Limitation. x_k can overflow $(10^5, 10^{28}, \dots)$ before returning to the boundary, so the target $x_{n+1} = 0$ becomes numerically unfindable.

Visual: Regular Shooting



Regular shooting sweeps the initial slope; Prüfer shooting sweeps the initial phase.

Prüfer Substitution: Phase and Amplitude

Change coordinates from Cartesian $(x_k, \Delta_h x_k)$ to polar (μ_k, θ_k):

$$x_k = \mu_k \sin\theta_k, \qquad \Delta_h x_k = \mu_k \cos\theta_k, \qquad \mu_k > 0, \quad 0 \le \theta_k < 2\pi.$$

Phase equation:

$$\sin(h_k \Delta_h \theta_k) = h_k\big(\cos\theta_k \cos\theta_{k+1} + \lambda h_k \cos\theta_k \sin\theta_{k+1} + \lambda \sin\theta_k \sin\theta_{k+1}\big).$$

Amplitude equation:

$$\Delta_h \mu_k = -\mu_k\left(\tfrac{h_k}{2}[(\Delta_h \sin\theta_k)^2 + (\Delta_h \cos\theta_k)^2] - \sin\theta_{k+1}\cos\theta_k + \lambda h_k \cos\theta_{k+1}\cos\theta_k + \lambda \cos\theta_{k+1}\sin\theta_k\right).$$

The phase equation is independent of the amplitude.

Why Phase Beats Amplitude

Amplitude μ_k

$$\mu_k = \sqrt{x_k^2 + (\Delta_h x_k)^2}$$

No bound: μ_k grows without limit whenever x_k does.

Phase θ_k

$$\theta_k = \operatorname{arccot}\left(\frac{\Delta_h x_k}{x_k}\right) \in (0, \pi).$$

Bounded: even when x_k blows up, θ_k stays in $(0, \pi)$.

Prüfer shooting tunes θ instead of x, so overflow of x_k never disturbs the target.

Eigenvalue Results ($n = 100$)

Uniform grid

k	$\lambda_k = k^2$	λ_k^{reg}	$\lvert\text{err}\rvert^{\text{reg}}$	λ_k^{Pr}	$\lvert\text{err}\rvert^{\text{Pr}}$
10	100	100.0157	1.57×10^{-2}	100.0000	7.31×10^{-11}
20	400	400.9019	9.02×10^{-1}	400.0000	1.93×10^{-12}
30	900	908.5147	8.51×10^{0}	900.0000	1.93×10^{-12}
40	1600	1634.7863	3.48×10^{1}	1600.0000	2.73×10^{-12}
50	2500	2569.4023	6.94×10^{1}	2500.0000	2.73×10^{-12}
60	3600	3597.2163	2.78×10^{0}	3600.0000	1.82×10^{-12}

Clustered grid ($\sigma_p = 0.7$, $\sigma_d = 0.3$)

k	$\lambda_k = k^2$	λ_k^{reg}	$\lvert\text{err}\rvert^{\text{reg}}$	λ_k^{Pr}	$\lvert\text{err}\rvert^{\text{Pr}}$
10	100	100.2768	2.77×10^{-1}	100.0000	7.31×10^{-11}
20	400	407.7564	7.76×10^{0}	400.0000	1.93×10^{-12}
30	900	840.3701	5.96×10^{1}	900.0000	1.93×10^{-12}
40	1600	1116.1373	4.84×10^{2}	1600.0000	2.73×10^{-12}
50	2500	1358.3034	1.14×10^{3}	2500.0000	2.73×10^{-12}
60	3600	1999.5097	1.60×10^{3}	3600.0000	1.82×10^{-12}

Graded grid (ratio $r = 1.3$)

k	$\lambda_k = k^2$	λ_k^{reg}	$\lvert\text{err}\rvert^{\text{reg}}$	λ_k^{Pr}	$\lvert\text{err}\rvert^{\text{Pr}}$
10	100	84.0273	1.60×10^{1}	100.0000	7.31×10^{-11}
20	400	1309.7276	9.10×10^{2}	400.0000	1.93×10^{-12}
23	529	2904.0758	2.38×10^{3}	529.0000	6.74×10^{-11}
30	900	—	—	900.0000	1.93×10^{-12}
40	1600	—	—	1600.0000	2.73×10^{-12}
60	3600	—	—	3600.0000	1.82×10^{-12}

Conclusion

- Amplitude is unbounded; regular shooting inherits that instability.
- Prüfer substitution decouples phase from amplitude and bounds the shooting variable in $(0, \pi)$.
- Prüfer-based shooting is grid-agnostic: it stays accurate on uniform, clustered, and graded grids, even where regular shooting breaks down.
- Zero counting via ϕ_n's $n-1$ interior zeros indexes eigenvalues reliably in both methods.

References

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