FUNCTION SPACES: DISCOVERY ACTIVITIES

SHALMALI BANDYOPADHYAY

Work together as a group. The goal of each activity is to figure out a concept by examining examples, not to be told the answer up front. Argue with each other. Try things. There are no wrong attempts.

1. What is a Function Space?

The Eight Candidates. For each set below, the elements are functions defined on the interval $[0, 1]$.

(1) All polynomials of degree at most 3.

(2) All polynomials of degree exactly 3.

(3) All continuous functions.

(4) All continuous functions f with $f(0) = 0$.

(5) All continuous functions f with $f(0) = 1$.

(6) All continuous, non-negative functions (i.e., $f(x) \geq 0$ for all $x \in [0, 1]$).

(7) The finite set $\{\sin x, \cos x, \sin 2x, \cos 2x\}$.

(8) All linear combinations $a \sin x + b \cos x$ where $a, b \in \mathbb{R}$.

Phase 1: Two Quick Tests. For each of the eight sets, answer:

(a) Can you find two functions in the set whose sum is also in the set?
(b) Can you find a function in the set whose scalar multiple (say, -1 times the function) is also in the set?
(c) Is the zero function (the function that equals 0 everywhere on $[0, 1]$) in the set?

For each set, decide: did it pass all three tests, or did at least one fail?

Phase 2: Look for the Pattern. Which sets passed all three tests? Which failed at least one?

Group the eight candidates into two columns: **passed everything** on the left, **failed something** on the right.

Phase 3: Synthesize. Write down, in your own words, the conditions a set of functions needs to satisfy to be a "function space."

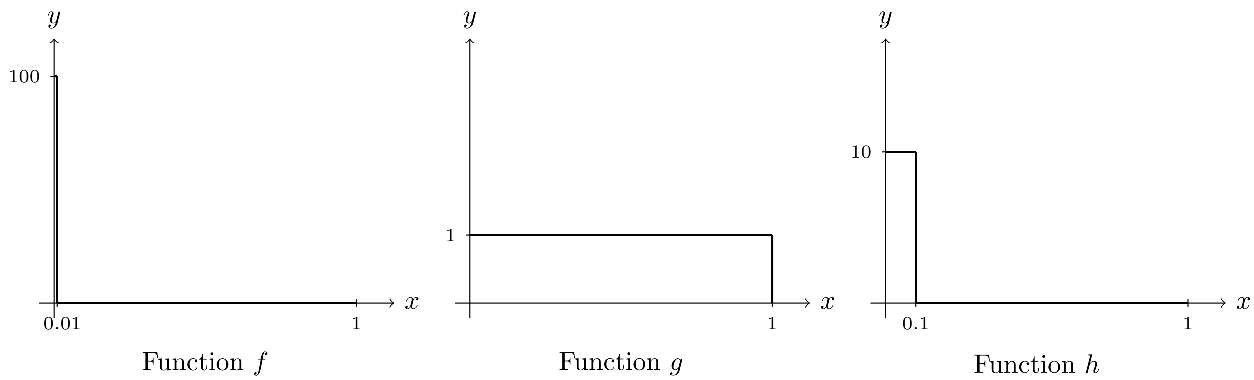
Now look at the vector space axioms from Dr. Nance's Tuesday lecture. How does your list compare?

Phase 4: Extend. Come up with one more example of a function space and one more non-example. Be ready to explain why each one passes or fails the tests.

2. Discovering the Concept of Size

Below are three functions f, g, and h defined on the interval $[0,1]$*:*

- $f(x) = \begin{cases} 100 & \text{if } 0 \le x \le 0.01 \\ 0 & \text{if } 0.01 < x \le 1 \end{cases}$
- $g(x) = 1$ for all $x \in [0,1]$.
- $h(x) = \begin{cases} 10 & \text{if } 0 \le x \le 0.1 \\ 0 & \text{if } 0.1 < x \le 1 \end{cases}$



Phase 1: Which Function is the Biggest? Without doing any calculations, just by looking at the three graphs: *which function would you say is the biggest? Justify your answer.*

Compare your reasoning with the others in your group. Do you all agree? If not, what is the source of the disagreement?

Phase 2: Different Ways to Measure "Size". Your group probably disagreed in Phase 1 because there is more than one reasonable way to measure the "size" of a function. Come up with at least three different ways one could assign a single non-negative number to a function $f : [0, 1] \to \mathbb{R}$ that captures some notion of its size or magnitude.

Write down each measurement scheme precisely as a formula or rule.

Phase 3: Compute and Compare. For each measurement scheme your group proposed in Phase 2, compute the "size" of each of f, g, and h. Record your results in the table below.

Measurement scheme	**Size of** f	**Size of** g	**Size of** h

Looking at your table: do all your measurement schemes agree on which function is the biggest? If they disagree, can you find one scheme under which f is the biggest, another under which g is the biggest, and another under which they appear the same?

Phase 4: Looking for Patterns. Two natural measurements of size are the *maximum value* of the function (which captures peak behavior) and the *area under the curve* (which captures total accumulation). These correspond to two extremes.

Can you propose a family of measurements that interpolates between these two extremes? That is, a family parametrized by a number p such that small p behaves like "area" and large p behaves like "peak"?

Hint: what happens if you raise $|f(x)|$ *to a power before integrating?*