Spectral Analysis of Discrete Non-Uniform Sturm-Liouville Problem

UT Martin | Bailyn Hall

Collaborators: Shalmali Bandyopadhyay, Jacob Blazejewski, Kimsear Lor

NREUP 2026

THE UNIVERSITY OF TENNESSEE MARTIN

Our Problem: Prüfer Transformation

Our Problem at a Glance

$$\begin{cases} \Delta_h\left(r_k \Delta_h x_k\right) + \left(\lambda w_k - q_k\right) x_{k+1} = 0 \quad \text{for } k = 0, \ldots, n-1 \\ x_0 = x_{n+1} = 0 \end{cases} \tag{SL-BVP}$$

$\mathbb{T} = t_0 < t_1 < t_2 < \ldots < t_{n-1}$ non-uniform grid; $h_k = t_{k+1} - t_k$

- $\Delta_h x_k = \dfrac{x_{k+1} - x_k}{h_k}$ vs. $\Delta x_k = x_{k+1} - x_k$
- $r_k, w_k : \mathbb{N} \to \mathbb{R}^+$, $q_k : \mathbb{N} \to \mathbb{R} \to$ weight functions
- SL-BVP $\underbrace{\equiv}_{\substack{\lambda \in \mathbb{R} \\ \text{parameter}}}$ EVP/Spectral Problem
- $r_k \Delta_h x_k$ is quasi-derivative $\to$ self-adjoint

$\boxed{\text{Prüfer} \equiv \text{Cartesian} \to \text{Polar}}$

Bohner–Došlý (2001)
[Introduced discrete Prüfer]

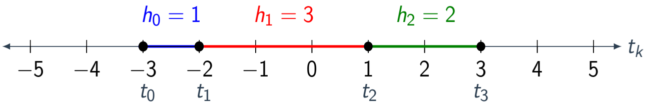
$$\begin{cases} \Delta\left(r_k \Delta x_k\right) + p_k x_{k+1} = 0 \\ x_0 = x_{n+1} = 0 \end{cases}$$

Çetinkaya–Er–Menken (2026)

$$\begin{cases} \Delta\left(r_k \Delta x_k\right) + (\lambda w_k - q_k) x_k = 0 \\ x_0 = x_{n+1} = 0 \end{cases}$$

where $k = 0, \ldots, n-1$
$\Delta x_k = x_{k+1} - x_k$

Our work



$$\begin{cases} \Delta_h\left(r_k \Delta_h x_k\right) + \left(\lambda w_k - q_k\right) x_{k+1} = 0 \\ x_0 = x_{n+1} = 0 \end{cases}$$

Recall: $k = 0, \ldots, n-1$

$$\Delta_h x_k = \frac{x_{k+1} - x_k}{t_{k+1} - t_k} = \frac{x_{k+1} - x_k}{h_k}$$

Note: $h_k > 0$ since $\{t_k\}$ is a strictly increasing sequence

The Goal of Our Problem

Goal: build the full spectrum of EVs and cor. EFs

$\lambda_1 < \lambda_2 < \cdots < \lambda_n,$ all real

Existence

each λ_i exists

Realness

each λ_i is real

Ordering

$\lambda_1 < \lambda_2 < \cdots$

Exactly n

eigenvalues

Together

complete, ordered spectrum

Why does any of this matter?

Discrete Sturm-Liouville Problem

- quantum mechanics, vibration, signals and systems, heat transfer and diffusion, fractional calculus, . . . , **SO MUCH MORE!**

Gap in literature

- Non-uniform, discrete Sturm-Liouville problem.

Non-uniformity

Non-uniformity of data reporting shows up everywhere in nature—in meterology (uneven reporting dates), ecology (seasonal data reporting), signals and systems (event-based activation/motion detection).

Discrete Pr̆ufer Substitution for Non-Uniform Step-size

Take $\lambda w_k - q_k = P_k$.

$$\underbrace{\begin{cases} \Delta_h(r_k \Delta_h x_k) + P_k x_{k+1} = 0 \\ x_0 = x_{n+1} = 0 \end{cases}}_{\text{2}^{\text{nd}}\text{ order BVP difference equation}} \quad \underset{\mu_k \text{ amp., } \theta_k \text{ phase}}{\overset{\text{Prüfer Substitution}}{\Longleftrightarrow}} \quad \underbrace{\begin{cases} x_k = \mu_k \sin \theta_k, \\ r_k \Delta_h x_k = \mu_k \cos \theta_k. \end{cases}}_{\text{Prüfer System}}$$

$$\Updownarrow$$

$$\underbrace{\begin{cases} \text{amplitude: } \Delta_h \mu_k = F\left(\theta_k\left(\lambda\right), \lambda, \mu_k, r_k, w_k, q_k\right), \ \text{IC: } \mu_0\left(\lambda\right) > 0 \\ \text{phase: } \sin\left(h_k \Delta_h \theta_k\right) = G\left(\theta_k\left(\lambda\right), \lambda, r_k, w_k, q_k\right), \ \text{IC: } \theta_0\left(\lambda\right) = 0, \ h_0 \Delta_h \theta_0\left(\lambda\right) > 0 \end{cases}}_{\text{system of 1}^{\text{st}}\text{ order IVP}}$$

Phase equation completely decoupled from amplitude! $\Rightarrow$ Track one quantity $(\theta_k(\lambda))$

Definition (Self-Adjointness)

- $u, v : \mathbb{C} \to \mathbb{C}$ & $\langle \cdot, \cdot \rangle$ complex inner product
- Linear operator $\mathcal{A}$ is self adjoint $\Leftrightarrow \langle \mathcal{A}u, v\rangle = \langle u, \mathcal{A}v\rangle$

$$\text{SL-BVP} \xleftrightarrow[\Delta_h(r_k \Delta_h x_k) \equiv \mathcal{L},\ \mathcal{A} \equiv -\mathcal{L}+Q]{\mathcal{W}\equiv\{w_k\}\ \&\ Q\equiv\{q_k\}} \text{Linear Operator Equation: } \mathcal{A}x = \lambda \mathcal{W}x$$

✓ property of quasi-derivative $\Rightarrow \mathcal{A}$ self adj.

✓ $w_k : \mathbb{R} \to \mathbb{R}^+ \Rightarrow \mathcal{W}$ self adj.

$$\Rightarrow \underbrace{\lambda\langle \mathcal{W}x, x\rangle} = \langle \lambda \mathcal{W}x, x\rangle = \langle \mathcal{A}x, x\rangle = \langle x, \mathcal{A}x\rangle = \langle x, \lambda \mathcal{W}x\rangle = \bar{\lambda}\langle x, \mathcal{W}x\rangle = \underbrace{\bar{\lambda}\langle \mathcal{W}x, x\rangle}$$

$$\Rightarrow (\lambda - \bar{\lambda})\langle \mathcal{W}x, x\rangle = 0$$

$$\Downarrow$$

EV's are real (if they exist). (1/4)

Discrete Derivative of Phase Eq.

Assumptions

- Choose r_k such that $X = h_k \cos\theta_k + r_k \sin\theta_k \neq 0$.
- $\theta_k \notin \pi\mathbb{Z}^+$ for $k = 1, \ldots, n$.

Discrete Derivative of Phase Equation

- Recall $\sin\left(h_k \Delta_h \theta_k\right) = G\left(\theta_k\left(\lambda\right), \lambda, r_k, w_k, q_k\right)$.
- Use $h_k \Delta_h \theta_k = \cancel{h_k} \underbrace{\frac{\theta_{k+1} - \theta_k}{\cancel{h_k}}}_{\text{definition of discrete deriv.}} = \theta_{k+1} - \theta_k$.
- Use trig identity $\sin\left(\theta_{k+1} - \theta_k\right) = \sin\theta_{k+1}\cos\theta_k - \sin\theta_k\cos\theta_{k+1}$ and rearrange terms.
- Divide by $\sin\theta_{k+1}$ & X; both $\neq 0$ from Assumption.
- $\frac{\cos\theta_{k+1}}{\sin\theta_{k+1}} = \cot\theta_{k+1} = \frac{Y}{X}$

Proposition: Monotonicity

Proposition 1 (Monotonicity)

◇ $\lambda \mapsto \theta_k(\lambda)$ is the unique solution to phase equation with I.C.

$$\boxed{\theta_k(\lambda) \text{ is strictly increasing on } \lambda.}$$

Process

$$\underbrace{\theta_{k+1} = \theta_k + \operatorname{arccot}\left(\frac{N(\theta_k(\lambda), \lambda)}{D(\theta_k(\lambda), \lambda)}\right)}_{\text{recursive relation}}$$

$$\frac{\partial}{\partial \lambda}(\text{LHS \& RHS}) \implies \text{multivari. chain rule} \implies \text{algebra} \implies \text{induc.} \implies > 0$$

Proposition: Eigenfunction Characterization

$$\text{SL-BVP} \xLeftrightarrow{\text{Prüfer Substitution}} \begin{array}{c} 1^{\text{st}} \text{ order D.E.} \\ (\text{amplitude} + \text{phase}) \end{array} \xLeftrightarrow{\text{Impose Initial Condition}} \text{system of IVP}$$

Proposition 2 (Eigenfunction Characterization)

- $\mu_k(\lambda)$ sol. to amplitude
- I.C. $\mu_0 > 0$
- $\theta_k(\lambda)$ sol. to phase
- I.C. $\theta_0(\lambda) = 0$

$$\boxed{\lambda \text{ is an eigenvalue of SL-BVP} \iff \theta_{n+1}(\lambda) = m\pi \text{ for some } m \in \mathbb{Z}^+}$$

$$\boxed{x_k = \mu_k(\lambda) \sin\theta_k(\lambda) \ \text{ is the corresponding eigenfunction of } \lambda}$$

Process

($\Rightarrow$) Assume λ eigenvalue $\implies$ Use def. of eigenvalue $\implies$ $\theta_{n+1}(\lambda)$ must be multiple of π

($\Leftarrow$) $\theta_{n+1}(\lambda)$ is multiple of π $\implies$ Use Prufer Substitution $\implies$ $\{x_k\}_{k=0}^{n} \not\equiv 0$ $\implies$ λ EV

Ordering

To Show

$$m_1 < m_2 \iff \lambda_{m_1} < \lambda_{m_2}$$

($\Rightarrow$): by contradiction

Suppose $m_1 < m_2$ and $\lambda_{m_1} \not< \lambda_{m_2}$.

- $\Rightarrow \theta_{n+1}(\lambda_{m_1}) \not< \theta_{n+1}(\lambda_{m_2})$
- EF Characterization
  $\Rightarrow \theta_{n+1}(\lambda_{m_i}) = m_i\pi, \ \ i = 1, 2$
- $m_1\pi \not< m_2\pi \Rightarrow m_1 \not< m_2$ contradiction!

($\Leftarrow$): direct

Suppose $\lambda_{m_1} < \lambda_{m_2}$.

- Monotonicity of Phase
  $\Rightarrow \theta_{n+1}(\lambda_{m_1}) < \theta_{n+1}(\lambda_{m_2})$
- EF Characterization
  $\Rightarrow \theta_{n+1}(\lambda_{m_i}) = m_i\pi, \ \ i = 1, 2$
- $m_1\pi < m_2\pi \Rightarrow m_1 < m_2$

$$m_1 < m_2 < \ldots < m_i \implies \theta_{n+1}(\lambda_{m_1}) < \theta_{n+1}(\lambda_{m_2}) < \ldots < \theta_{n+1}(\lambda_{m_i})$$

All of our eigenvalues are ordered! (2/4) This also implies they are unique.

$$\lambda_1 < \lambda_2 < \ldots < \lambda_i \tag{2/4}$$

Exactness

To Show

We have exactly n eigenvalues.

Process

- Put problem into matrix form.
- Use definition of determinant and characteristic polynomial.
- Fundamental Thm of Alg. $\Rightarrow$ at most $n \in \mathbb{R}$ eigenvalues (counting multiplicity)

$$\text{Uniqueness} \implies \boxed{\text{Exactly } n \in \mathbb{R} \text{ E.V.}} \tag{3/4}$$

Exactness

$$
\underbrace{\begin{bmatrix}
\frac{r_1}{h_1}+\frac{r_0}{h_0}+h_0 q_0 & -\frac{r_1}{h_1} & 0 & \cdots & 0 \\
\frac{r_1}{h_1} & \frac{r_2}{h_2}+\frac{r_1}{h_1}+h_1 q_1 & -\frac{r_2}{h_2} & \ddots & \vdots \\
0 & \frac{r_2}{h_2} & \frac{r_3}{h_3}+\frac{r_2}{h_2}+h_2 q_2 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & -\frac{r_{n-1}}{h_{n-1}} \\
0 & \cdots & 0 & \frac{r_{n-1}}{h_{n-1}} & \frac{r_n}{h_n}+\frac{r_{n-1}}{h_{n-1}}+h_{n-1} q_{n-1}
\end{bmatrix}}_{n\times n}
= \lambda \underbrace{\begin{bmatrix}
h_0 w_0 & 0 & \cdots & 0 \\
0 & h_1 w_1 & \cdots & 0 \\
\vdots & 0 & \ddots & \vdots \\
0 & 0 & \cdots & h_{n-1} w_{n-1}
\end{bmatrix}}_{n\times n}
$$

Exactness

$$B - \lambda D = M =$$

$$\underbrace{\begin{bmatrix} \frac{r_1}{h_1}+\frac{r_0}{h_0}+h_0(q_0-\lambda w_0) & -\frac{r_1}{h_1} & 0 & \cdots & 0 \\ \frac{r_1}{h_1} & \frac{r_2}{h_2}+\frac{r_1}{h_1}+h_1(q_1-\lambda w_1) & -\frac{r_2}{h_2} & \ddots & \vdots \\ 0 & \frac{r_2}{h_2} & \frac{r_3}{h_3}+\frac{r_2}{h_2}+h_2(q_2-\lambda w_2) & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\frac{r_{n-1}}{h_{n-1}} \\ 0 & \cdots & 0 & \frac{r_{n-1}}{h_{n-1}} & \frac{r_n}{h_n}+\frac{r_{n-1}}{h_{n-1}}+h_{n-1}(q_{n-1}-\lambda w_{n-1}) \end{bmatrix}}_{n \times n}$$

$$\det(M) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} M_{i,\sigma(i)},$$

$$\underbrace{P(\lambda) = C_n \lambda^n + C_{n-1}\lambda^{n-1} + \dots}_{\text{characteristic polynomial}}$$

Showed: $C_n \neq 0 \implies \deg P = n$

All Put Together: Spectral Theorem

Theorem (The Spectral Theorem)

- All of our eigenvalues are **real**. (1/4)
- The eigenvalues are **unique** and **ordered**. (2/4)
- We have **exactly** n eigenvalues that forms a **spectrum**. (3/4)
- ➠ Future work: The eigenvalues of our problem **exist**. (4/4)

Acknowledgments

Research Mentors Dr. Shalmali Bandyopadhyay & Dr. Ezra Nance

Research Advisors Dr. Shalmali Bandyopadhyay & Dr. Jacob Blazejewski (Appalachian State University)

UTM Department of Mathematics & Statistics at UT Martin (Lunches, Research resources, Feedback on weekly presentations)

MAA-NSF Mathematical Association of America and NSF-DMS 2308688

THE UNIVERSITY OF TENNESSEE MARTIN

Questions?

Email: bhall76@ut.utm.edu