Prüfer Transformation for a Sturm Liouville Type Equation

Advisor: S. Bandyopadhyay   Presenters: Kimsear Lor & Bailyn Hall
Department of Mathematics & Statistics, University of Tennessee at Martin

The Sturm–Liouville Problem

The general Sturm–Liouville (SL) equation:

$$DHi-(p(x)\,y')'+q(x)\,y=\lambda\,w(x)\,y,\quad x\in[a,b]$$

with separated boundary conditions:

$$\alpha_1 y(a)+\alpha_2 y'(a)=0,\qquad \beta_1 y(b)+\beta_2 y'(b)=0$$

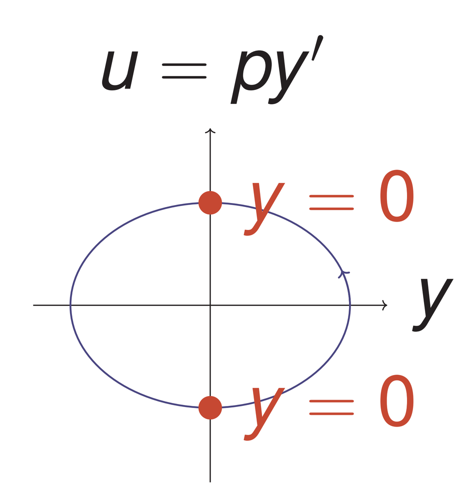
where $p(x)>0$, $w(x)>0$, $q(x)$ continuous on $[a,b]$.

The Phase Plane

Set $u=p(x)y'$. The SL equation is equivalent to:

$$y'=\frac{u}{p(x)},\qquad u'=\big(q(x)-\lambda w(x)\big)\,y$$

The solution $(y(x),u(x))$ traces a curve in the (y,u) plane.

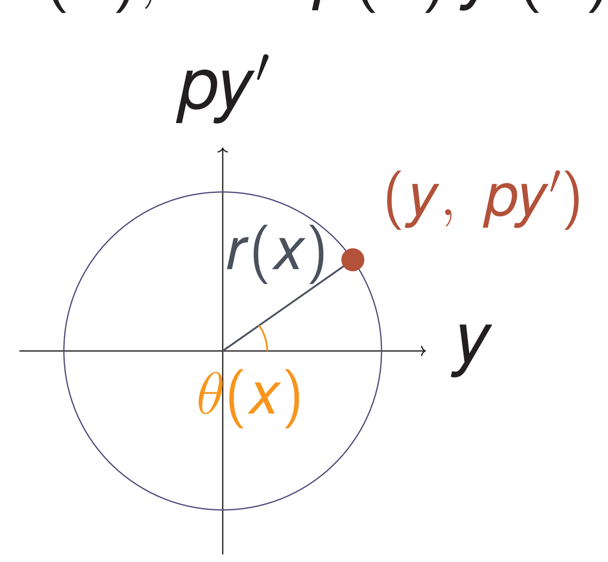


$y(x_0)=0 \iff$ the trajectory crosses the u-axis.

The Prüfer Substitution

Switch to polar coordinates in the (y,u) plane:

$$y(x)=r(x)\sin\theta(x),\qquad p(x)\,y'(x)=r(x)\cos\theta(x)$$



As x increases the point $(y,\ py')$ revolves around the origin.

References

A. Zettl, *Sturm–Liouville Theory*, Mathematical Surveys and Monographs, Vol. 121, American Mathematical Society, 2005.

Differentiating the Substitution

Differentiate $y=r\sin\theta$ and use $py'=r\cos\theta$:

Equation A:

$$r'\sin\theta+r\theta'\cos\theta=\frac{r}{p}\cos\theta$$

Differentiate $py'=r\cos\theta$ and substitute the SL equation:

Equation B:

$$r'\cos\theta-r\theta'\sin\theta=-(q-\lambda w)\,r\sin\theta$$

Deriving the Prüfer System

Eliminate r': multiply A by $\cos\theta$, B by $\sin\theta$, subtract:

$$r\theta'(\cos^2\theta+\sin^2\theta)=\frac{r}{p}\cos^2\theta+(\lambda w-q)\,r\sin^2\theta$$

Use $\cos^2\theta+\sin^2\theta=1$ and divide by r:

$$\boxed{\theta'(x)=\frac{\cos^2\theta}{p(x)}+\big(\lambda w(x)-q(x)\big)\sin^2\theta}$$

Eliminate θ': multiply A by $\sin\theta$, B by $\cos\theta$, add, divide by r:

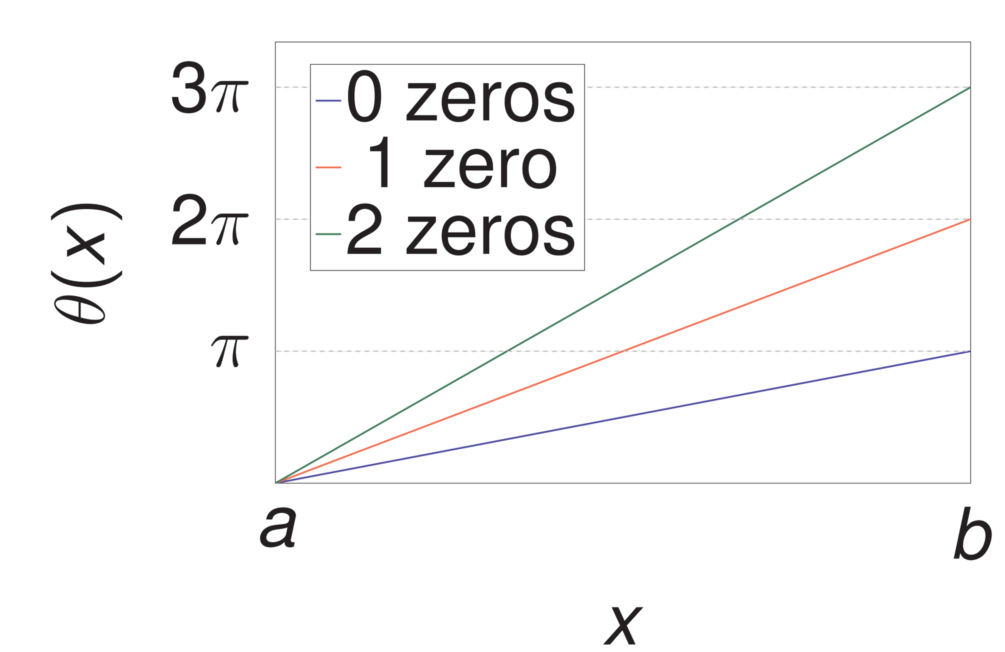
$$\boxed{r'(x)=r(x)\left(\frac{1}{p(x)}-\big(q(x)-\lambda w(x)\big)\right)\sin\theta\cos\theta}$$

Consequences of the Prüfer System

θ' is independent of r – solve the θ equation alone.

$$y(x_0)=0\iff\theta(x_0)=n\pi$$

The number of zeros of y on $[a,b]$ equals the number of multiples of π that θ crosses.



E-Mail: sbanyo5@ut.utm.edu