

# Phase, Amplitude, and Spectrum

A Prüfer Approach to Non-Self-Adjoint Sturm–Liouville Problems

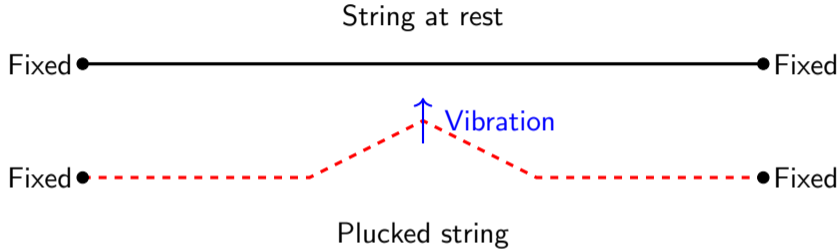
Shalmali Bandyopadhyay

University of Tennessee at Martin

Joint work with F. Ayça Çetinkaya (UTC) and Tom Cuchta (Marshall)

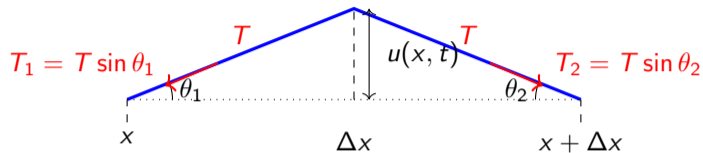
Central Michigan University Mathematics Colloquium

# What happens when you pluck a guitar string?



- The string vibrates in patterns called **modes**
- Each mode has a specific frequency (pitch)
- **Mathematics describes these vibrations precisely**

# The Wave Equation: Physical Derivation



For small  $\theta$ :

$$\sin \theta \approx \tan \theta = \frac{\partial u}{\partial x}$$

$$\begin{aligned}\rho \Delta x \frac{\partial^2 u}{\partial t^2} &= T \sin \theta_2 - T \sin \theta_1 \\ \rho \Delta x \frac{\partial^2 u}{\partial t^2} &= T \left[ \left. \frac{\partial u}{\partial x} \right|_{x+\Delta x} - \left. \frac{\partial u}{\partial x} \right|_x \right] \\ \rho \frac{\partial^2 u}{\partial t^2} &= \frac{T \left[ \left. \frac{\partial u}{\partial x} \right|_{x+\Delta x} - \left. \frac{\partial u}{\partial x} \right|_x \right]}{\Delta x} \\ \lim_{\Delta x \rightarrow 0} \rho \frac{\partial^2 u}{\partial t^2} &= \lim_{\Delta x \rightarrow 0} \frac{T \left[ \left. \frac{\partial u}{\partial x} \right|_{x+\Delta x} - \left. \frac{\partial u}{\partial x} \right|_x \right]}{\Delta x}\end{aligned}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c = \sqrt{\frac{T}{\rho}}$$

# Separation of Variables: PDE $\rightarrow$ ODE

Try solutions of the form:  $u(x, t) = X(x)T(t)$

$$\frac{\partial^2 u}{\partial t^2} = X(x)T''(t)$$

$$\frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$$

Substitute into  $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ ,  $X(x)T''(t) = c^2 X''(x)T(t)$

Divide by  $X(x)T(t)$ ,

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)}$$

left side depends only on  $t$  and right side only on  $x$ , both must equal a constant! (say  $-\lambda$ ), where  $\lambda > 0$

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

# Solving ODEs

**Spatial:**

$$X''(x) + \lambda X(x) = 0$$

$$X(0) = X(L) = 0$$

Characteristic equation:

$$r^2 + \lambda = 0 \implies r = \pm i\sqrt{\lambda}$$

$$\text{G.S.: } X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \lambda_n = \frac{n^2\pi^2}{L^2}$$

**Time:**

$$T''(t) + \lambda_n c^2 T(t) = 0$$

Characteristic equation:

$$r^2 + \lambda_n c^2 = 0 \implies r = \pm i\sqrt{\lambda_n} c$$

$$\text{G.S.: } T_n(t) = A_n \cos(\omega_n t) + B_n \sin(\omega_n t)$$

$$\omega_n = \frac{n\pi c}{L}$$

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)]$$

$X_n(x)$  is the **pattern** of the vibration — the mode shape determines the **note**.

$T_n(t)$  is the **speed** of that pattern — the frequency determines the **pitch**.

# Space Equation Revisited

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(L) = 0$$

Note that  $X(x) \equiv 0$  is always a solution. We look for **nontrivial** solutions.

General solution:  $X(x) = A \cos(\sqrt{\lambda} x) + B \sin(\sqrt{\lambda} x)$

$$X(0) = 0 \implies A = 0 \implies X(x) = B \sin(\sqrt{\lambda} x)$$

$$X(L) = 0 \implies B \sin(\sqrt{\lambda} L) = 0$$

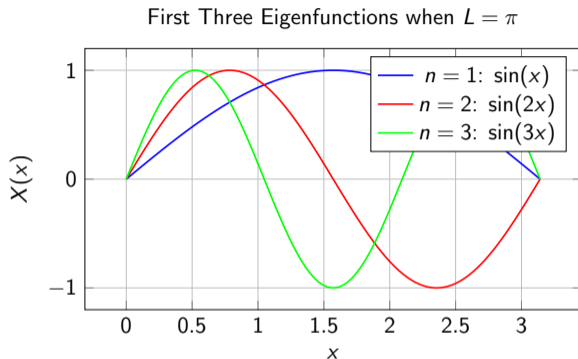
For nontrivial solution  $B \neq 0$ , so  $\sin(\sqrt{\lambda} L) = 0$ , which gives:

$$\sqrt{\lambda} L = n\pi \implies \boxed{\lambda_n = \frac{n^2 \pi^2}{L^2}, \quad n = 1, 2, 3, \dots}$$

$\lambda_n$  are the **eigenvalues**,  $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$  are the **eigenfunctions**.

This is an **eigenvalue problem (EVP)**.

# Visualizing the First Few Modes



- Mode 1: Fundamental frequency (lowest pitch)
- Mode 2: First harmonic (octave higher)
- Mode 3: Second harmonic
- Higher modes = higher frequencies = higher pitches
- $n^{\text{th}}$  eigenfunction has  $n - 1$  zeros — characterize the oscillatory behavior of the solution

# Self-Adjointness of the EVP

Our EVP  $X''(x) + \lambda X(x) = 0$  can be written as:

$$\mathcal{A}X = \lambda X, \quad \mathcal{A} = -\frac{d^2}{dx^2}, \quad \mathcal{A}u = -u''$$

The operator  $\mathcal{A}$  is **self-adjoint** if:

$$\langle \mathcal{A}u, v \rangle = \langle u, \mathcal{A}v \rangle, \quad \text{where } \langle u, v \rangle = \int_0^L u(x)v(x) dx$$

**Claim:**  $\mathcal{A}$  is self-adjoint for our problem  $X(0) = X(L) = 0$ .

**Proof:** Integrate by parts twice:

$$\int_0^L (-u'')v dx = \underbrace{-u'v \Big|_0^L}_{=0} + \int_0^L u'v' dx = \underbrace{-uv' \Big|_0^L}_{=0} + \int_0^L u(-v'') dx$$

Therefore:

$$\langle \mathcal{A}u, v \rangle = \langle u, \mathcal{A}v \rangle \quad \checkmark$$

The boundary conditions  $u(0) = u(L) = 0$  make the boundary terms vanish.

# EVP → The Sturm–Liouville Problem

## Differential Equation:

$$X''(x) + \lambda X(x) = 0$$

$$-[1 \cdot X']' + 0 \cdot X = \lambda \cdot 1 \cdot X$$

$$\downarrow \quad 1 \rightarrow p(x), \quad 0 \rightarrow q(x), \quad 1 \rightarrow w(x), \quad X \rightarrow y$$

$$-[p(x)y']' + q(x)y = \lambda w(x)y$$

## Boundary Condition:

$$X(0) = X(L) = 0$$

$$\downarrow \quad X \rightarrow y, \quad 0 \rightarrow a, \quad L \rightarrow b$$

$$1 \cdot y(a) + 0 \cdot y'(a) = 0$$

$$1 \cdot y(b) + 0 \cdot y'(b) = 0$$

$$\downarrow \quad 1 \rightarrow \alpha_1, \quad 0 \rightarrow \alpha_2, \quad 1 \rightarrow \beta_1, \quad 0 \rightarrow \beta_2$$

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \quad \beta_1 y(b) + \beta_2 y'(b) = 0$$

## Types of boundary conditions:

- Neumann:  $y'(a) = 0, y'(b) = 0$  ( $\alpha_1 = 0, \alpha_2 = 1, \beta_1 = 0, \beta_2 = 1$ )
- Robin:  $\alpha_1 y(a) + \alpha_2 y'(a) = 0, \beta_1 y(b) + \beta_2 y'(b) = 0$
- **Dirichlet:**  $y(a) = 0, y(b) = 0$  ← **our focus** ( $\alpha_1 = 1, \alpha_2 = 0, \beta_1 = 1, \beta_2 = 0$ )

# Properties of Self-Adjoint SL Problems

For the **self-adjoint? (Ex.)** SL problem  $-[p(x)y']' + q(x)y = \lambda w(x)y$  with  $y(a) = y(b) = 0$ :

- Eigenvalues are **real**:

$$\lambda_n \in \mathbb{R} \quad \text{for all } n$$

- Eigenvalues form a discrete sequence:

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots \rightarrow \infty$$

- Eigenfunctions are **orthogonal** with respect to weight  $w(x)$ :

$$\int_a^b y_m(x) y_n(x) w(x) dx = 0, \quad m \neq n$$

- Eigenfunctions are **complete** — any  $f \in L^2([a, b])$  can be expanded:

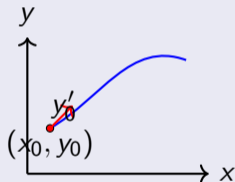
$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x)$$

- **Sturm oscillation theorem** — the  $n$ -th eigenfunction  $y_n$  has exactly  $n - 1$  interior zeros

# The SL Problem with Dirichlet Boundary: Hard to Solve

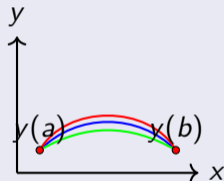
$$-[p(x)y']' + q(x)y = \lambda w(x)y, \quad x \in [a, b], \quad y(a) = y(b) = 0$$

## Initial Value Problem



- Given:  $y(x_0), y'(x_0)$
- **Unique solution** exists
- March forward step by step

## Boundary Value Problem



- Given:  $y(a), y(b)$
- May have 0, 1, or  $\infty$  solutions
- Global problem

# The Shooting Method: Basic Idea

## Converting BVP to IVP

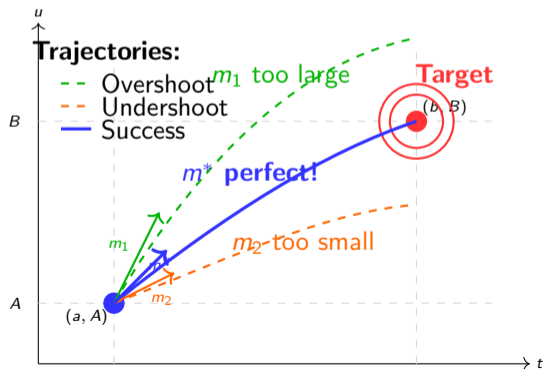
Instead of solving  $y'' + f(x, y, \lambda) = 0$  with  $y(a) = y(b) = 0$ :

**Step 1:** Guess an eigenvalue  $\lambda$

**Step 2:** Solve the IVP:  $y'' + f(x, y, \lambda) = 0$ ,  $y(a) = 0$ ,  $y'(a) = 1$

**Step 3:** Check if  $y(b) = 0$

**Step 4:** If not, adjust  $\lambda$  and repeat



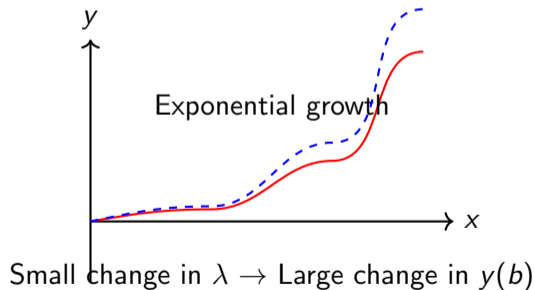
### Algorithm:

1. Start at  $(a, A)$
2. Guess slope  $m$
3. Solve IVP
4. Check if  $u(b) = B$
5. Adjust  $m$  if needed
6. Repeat until target hit

# Shooting Method Challenges

## Numerical Instabilities

- **Exponential growth:** Solutions can grow/decay exponentially
- **Loss of precision:** Small errors amplify dramatically
- **Poor conditioning:** Tiny changes in  $\lambda$  cause huge changes in  $y(b)$

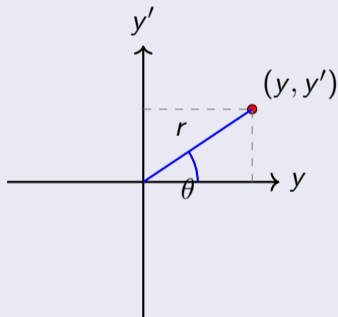


**Result:** Traditional shooting can fail spectacularly!

# The Revolutionary Idea: Heinz Prüfer (1926)

## From Cartesian to Polar

Instead of tracking  $(y, y')$  directly, use **polar coordinates**:



## The Prüfer Transformation

$$y(x) = r(x) \sin(\theta(x))$$

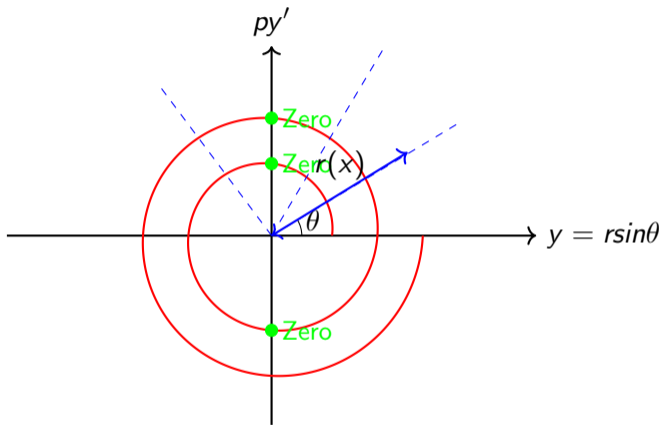
$$y'(x) = r(x) \cos(\theta(x))$$

where:

- $r(x) \geq 0$  is the **amplitude**
- $\theta(x)$  is the **phase**

Convert a second-order ODE into a geometric problem!

# Why This Perspective is Powerful



**Zeros of  $y(x)$  occur exactly when  $\theta(x) = n\pi$  (i.e., when the trajectory crosses the  $py'$ -axis)**

# The Prüfer Transformation

The **Prüfer substitution**:

$$y(x) = r(x) \sin \theta(x), \quad p(x)y'(x) = -r(x) \cos \theta(x)$$

Substituting into  $-[p(x)y']' + q(x)y = \lambda w(x)y$  gives the **Prüfer system**:

## Phase Equation

$$\theta'(x) = -\frac{\cos^2 \theta}{p(x)} + (q(x) - \lambda w(x)) \sin^2 \theta$$

## Amplitude Equation

$$r'(x) = r(x) \left( \frac{1}{p(x)} - q(x) + \lambda w(x) \right) \sin \theta \cos \theta$$

## Key observation

The phase equation is **independent of**  $r$ . Solve for  $\theta$  first. Then  $r$  follows — and  $r(x) > 0$  always.

# The Phase Equation: From $a$ to $b$

Starting from  $y(a) = 0$ :  $y(a) = 0 \implies \sin \theta(a) = 0 \implies \theta(a) = 0$

Solve the phase equation **forward** from  $a$  to  $b$ :

$$\theta'(x) = -\frac{\cos^2 \theta}{p(x)} + (q(x) - \lambda w(x)) \sin^2 \theta, \quad \theta(a) = 0$$

This is an **initial value problem** — march forward stably from  $a$  to  $b$ .

When we reach  $x = b$ , since  $y(b) = r(b) \sin \theta(b)$  and  $r(b) > 0$ :

$$y(b) = 0 \iff \sin \theta(b) = 0 \iff \theta(b) = n\pi \text{ for some } n \in \mathbb{N}$$

$$\theta(b) = n\pi$$

- $y(b) = 0$  ✓
- Boundary condition satisfied
- $\lambda$  is an **eigenvalue**
- $y$  has  $n - 1$  interior zeros

$$\theta(b) \neq n\pi$$

- $y(b) \neq 0$
- Boundary condition **not** satisfied
- $\lambda$  is **not** an eigenvalue
- Adjust  $\lambda$  and repeat

# What We Have Done So Far

- Started with a **vibrating string** — led naturally to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

- Separation of variables gave us a **Sturm–Liouville problem**:

$$-[p(x)y']' + q(x)y = \lambda w(x)y, \quad y(a) = y(b) = 0$$

- This operator is **self-adjoint** — guarantees real eigenvalues, orthogonal eigenfunctions, Sturm oscillation theorem
- The **Prüfer transformation** converts this into a first order IVP for  $\theta$ :

$$\theta'(x) = -\frac{\cos^2 \theta}{p(x)} + (q(x) - \lambda w(x)) \sin^2 \theta, \quad \theta(a) = 0$$

- Marching  $\theta$  forward from  $a$  to  $b$  — eigenvalues detected when  $\theta(b) = n\pi$

**But what if the operator is no longer self-adjoint?**

# A Non-Self-Adjoint Example

Consider:  $-y'' + y' = \lambda y$ ,  $y(0) = y(1) = 0$

The corresponding operator is  $\mathcal{B}y = -y'' + y'$ . Compare with the self-adjoint case:

$$Ay = -y'' \quad \longrightarrow \quad By = -y'' + y'$$

**Claim:**  $\mathcal{B}$  is **not** self-adjoint, i.e.,  $\langle \mathcal{B}u, v \rangle \neq \langle u, \mathcal{B}v \rangle$ .

**Proof:**

$$\begin{aligned}\langle \mathcal{B}u, v \rangle &= \int_0^1 (-u'' + u')v \, dx \\ &= \int_0^1 u(-v'') \, dx + \underbrace{[uv]_0^1}_{=0} - \int_0^1 uv' \, dx \\ &= \int_0^1 u(-v'' - v') \, dx\end{aligned}$$

But  $\langle u, \mathcal{B}v \rangle = \int_0^1 u(-v'' + v') \, dx$ . Since  $-v'' - v' \neq -v'' + v'$ :

$$\langle \mathcal{B}u, v \rangle \neq \langle u, \mathcal{B}v \rangle \quad \checkmark$$

# What Self-Adjointness Gives Us — And What Happens Without It

The Prüfer substitution is purely mechanical — it does not need self-adjointness. But the **conclusions** from the phase equation depend heavily on it:

## Classical Self-Adjoint Setting

- Zeros track eigenvalues via Sturm oscillation theorem
- $n$ -th eigenfunction has exactly  $n - 1$  zeros
- Monotonicity of zeros
- Comparison theorems
- Eigenvalue bounds via Rayleigh quotient

## Without Self-Adjointness

The geometric picture is there but the theory collapses:

- Sturm oscillation theorem **fails**
- No guarantee on zeros
- Comparison theorems **fail**
- Rayleigh quotient not directly applicable

## What We Establish

- |                           |                               |
|---------------------------|-------------------------------|
| • Monotonicity of zeros ✓ | • Phase criterion ✓           |
| • Eigenvalue bounds ✓     | • Sturm oscillation theorem ✗ |

# Our Problem

Recall  $\mathcal{B}y = -y'' + y'$  was non-self-adjoint due to the extra  $y'$  term. More generally, replacing  $y'$  with the **quasi-derivative**  $y' + s(x)y$  leads to:

$$-[\rho(x)(y'(x) + s(x)y(x))]' + s(x)\rho(x)(y'(x) + s(x)y(x)) + q(x)y(x) = \lambda\omega(x)y(x)$$

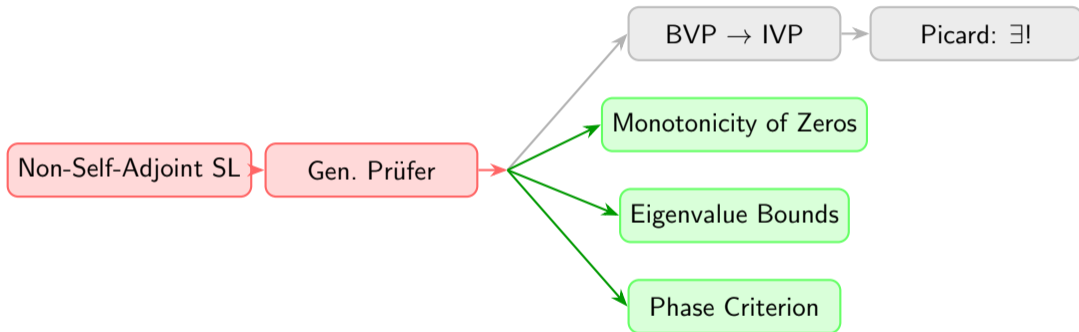
with Dirichlet boundary conditions  $y(a) = y(b) = 0$ , where  $p, s, q, \omega \in C[a, b]$ ,  $p(x) > 0$  and  $\omega(x) > 0$  on  $[a, b]$ .

## Key Feature: The Quasi-Derivative

$$D_s y := y'(x) + s(x)y(x)$$

- Classical SL uses  $y' \rightarrow$  self-adjoint
- Our problem uses  $D_s y = y' + s(x)y \rightarrow$  **non-self-adjoint**

# Road Map



# The Generalized Prüfer Transformation

Recall the classical Prüfer substitution:

$$y(x) = r(x) \sin \theta(x), \quad p(x)y'(x) = -r(x) \cos \theta(x)$$

For our equation with quasi-derivative  $D_s y = y' + s(x)y$ , we adapt the substitution:

$$y(x) = r(x) \sin \theta(x), \quad p(x)(y'(x) + s(x)y(x)) = -r(x) \cos \theta(x)$$

## The Key Change

$$p(x)y'(x) \longrightarrow p(x)(y'(x) + s(x)y(x)) = p(x)D_s y$$

Replace  $y'$  with the quasi-derivative  $D_s y$  throughout. This is the natural substitution adapted to the structure of our equation.

The amplitude and phase are given by:

$$r^2(x) = y^2(x) + [p(x)(y'(x) + s(x)y(x))]^2$$
$$\theta(x) = \arctan \left( \frac{y(x)}{-p(x)(y'(x) + s(x)y(x))} \right)$$

# The Generalized Prüfer System — Same Old Story

Substituting into our equation gives the **generalized Prüfer system**:

## Phase Equation

$$\theta'(x) = -\frac{\cos^2 \theta}{p(x)} - 2s(x) \sin \theta \cos \theta + (q(x) - \lambda\omega(x)) \sin^2 \theta$$

## Amplitude Equation

$$r'(x) = r(x) \left[ -\frac{\cos \theta \sin \theta}{p(x)} - s(x) \sin^2 \theta + s(x) \cos^2 \theta - (q(x) - \lambda\omega(x)) \sin \theta \cos \theta \right]$$

## Key Observations

- Phase equation is **independent of  $r$**  — solve for  $\theta$  first
- $r(x) > 0$  always — zeros of  $y$  still tracked by  $\theta$  alone
- $y(x) = 0 \iff \sin \theta(x) = 0 \iff \theta(x) \in \pi\mathbb{Z}$

# Existence and Uniqueness: The IVP for $\theta$

The phase equation is an **initial value problem**:

$$\theta'(x) = f(x, \theta), \quad \theta(a) = 0$$

**Key question:** Does this IVP have a unique solution?

## Lipschitz Condition — Intuition

If two solutions  $\theta_1$  and  $\theta_2$  start close together, they **stay** close together:

$$|f(x, \theta_1) - f(x, \theta_2)| \leq \hat{L}|\theta_1 - \theta_2|$$

This controls how fast solutions can diverge — ruling out branching behavior.

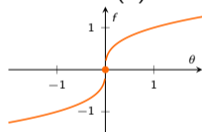
## Picard–Lindelöf Theorem

If  $f(x, \theta)$  is continuous in  $x$  and Lipschitz in  $\theta$ , then the IVP has a **unique solution** on  $[a, b]$ .

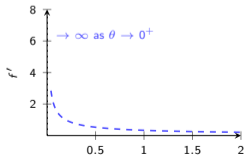
**Why is Lipschitz necessary?** Consider  $\theta' = \theta^{1/3}$ ,  $\theta(0) = 0$ . Two solutions exist:

$$\theta(x) \equiv 0 \quad \text{and} \quad \theta(x) = \left(\frac{2x}{3}\right)^{3/2}$$

**Function:**  $f(\theta) = \theta^{1/3}$



**Derivative:**  $f'(\theta) = \frac{1}{3}\theta^{-2/3}$



# Our Phase Equation is Lipschitz

$$\theta'(x) = -\frac{\cos^2 \theta}{p(x)} - 2s(x) \sin \theta \cos \theta + (q(x) - \lambda\omega(x)) \sin^2 \theta$$

$$f(x, \theta) = -\frac{\cos^2 \theta}{p(x)} - 2s(x) \sin \theta \cos \theta + (q(x) - \lambda\omega(x)) \sin^2 \theta$$

Compute  $\frac{\partial f}{\partial \theta}$ :  $\frac{\partial f}{\partial \theta} = \frac{\sin(2\theta)}{p(x)} - 2s(x) \cos(2\theta) + (q(x) - \lambda\omega(x)) \sin(2\theta)$

Using  $|\sin(2\theta)| \leq 1$  and  $|\cos(2\theta)| \leq 1$  and the triangle inequality:

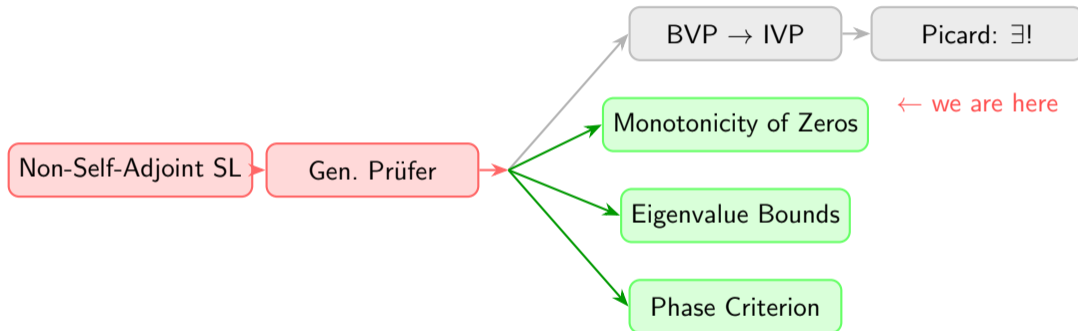
$$\left| \frac{\partial f}{\partial \theta} \right| \leq \frac{1}{p(x)} + 2|s(x)| + |q(x) - \lambda\omega(x)|$$

Taking supremum over  $[a, b]$  gives the Lipschitz constant:

$$\hat{L} = \sup_{x \in [a, b]} \left\{ \frac{1}{p(x)} + 2|s(x)| + |q(x) - \lambda\omega(x)| \right\} < \infty$$

$f$  is continuous in  $x$  and Lipschitz in  $\theta \implies$  by Picard–Lindelöf, the phase equation has a **unique solution** on all of  $[a, b]$ .

# Where are we on Road Map



# Monotonicity of Zeros

## Theorem

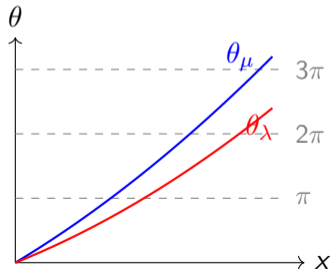
Let  $\mu > \lambda$ . Then the zeros of  $y_\mu$  satisfy:

$$x_k(\mu) < x_k(\lambda) \quad \text{for all } k$$

As  $\lambda$  increases, the zeros of  $y$  move **leftward**.

## Intuition:

- Larger  $\lambda \implies$  phase accumulates faster
- $\theta_\mu$  reaches each multiple of  $\pi$  **before**  $\theta_\lambda$
- Zeros occur earlier  $\implies$  shift left



# Monotonicity of Zeros: Proof

**Claim:** If  $\mu > \lambda$ , then  $x_k(\mu) < x_k(\lambda)$  for all  $k$ .

**Step 1: Phase equations.** Both  $\theta_\mu$  and  $\theta_\lambda$  satisfy their respective phase equations with  $\theta_\mu(a) = \theta_\lambda(a) = 0$ .

**Step 2: Compare right-hand sides.**

$$f_\mu(x, \theta) - f_\lambda(x, \theta) = (\lambda - \mu)\omega(x)\sin^2\theta \leq 0$$

since  $\mu > \lambda$ ,  $\omega(x) > 0$ , and  $\sin^2\theta \geq 0$ . Therefore  $f_\mu \leq f_\lambda$  everywhere.

**Step 3: Phase functions are decreasing.** At  $\theta = 0$ , we have  $\theta' = -\frac{1}{p(x)} < 0$ . So both  $\theta_\mu$  and  $\theta_\lambda$  are strictly decreasing through each multiple of  $\pi$ .

**Step 4:  $\theta_\mu$  decreases faster.** Since  $f_\mu \leq f_\lambda$ , we have  $\theta'_\mu \leq \theta'_\lambda$  — the function  $\theta_\mu$  is decreasing strictly faster than  $\theta_\lambda$ .

**Step 5: Comparison theorem.** Since  $\theta_\mu(a) = \theta_\lambda(a) = 0$  and  $\theta'_\mu \leq \theta'_\lambda$ , the ODE comparison theorem gives:

$$\theta_\mu(x) < \theta_\lambda(x) \quad \text{for all } x > a$$

**Step 6: Zeros shift left.** Zeros of  $y$  occur when  $\theta \in \pi\mathbb{Z}$ . Since  $\theta_\mu$  is strictly below  $\theta_\lambda$  and both are decreasing,  $\theta_\mu$  reaches each  $-k\pi$  at a smaller value of  $x$ :

$$x_k(\mu) < x_k(\lambda) \quad \text{for all } k \quad \square$$

# Eigenvalue Bounds — Why that Matters



- The natural frequency of the building  $\longleftrightarrow$  eigenvalue  $\lambda_n$
- Must stay in the **safe zone** — away from earthquake frequencies
- The **tuned mass damper** dissipates energy  $\longrightarrow$  operator is **non-self-adjoint**

Without solving the differential equation, can we find an explicit interval containing  $\lambda_n$ ?

# The Rayleigh Quotient

For the classical self-adjoint SL problem, multiply by  $y$  and integrate by parts to get:

$$R[y] = \frac{\int_a^b p(x)(y')^2 dx + \int_a^b q(x)y^2 dx}{\int_a^b \omega(x)y^2 dx}$$

The  $n$ -th eigenvalue is characterized by the **minimax principle**:

$$\lambda_n = \min_{\substack{S \subset H_0^1(a,b) \\ \dim S = n}} \max_{\substack{v \in S \\ v \neq 0}} R[v]$$

## Problem for Our Equation

Multiplying our non-self-adjoint equation by  $y$  and integrating gives:

$$R[y] = \frac{\int_a^b p(x)(y' + s(x)y)^2 dx + \int_a^b q(x)y^2 dx}{\int_a^b \omega(x)y^2 dx}$$

The minimax principle requires self-adjointness — it **cannot** be applied directly here.

# Transforming to a Self-Adjoint Problem

**Game changer:** change of variables:  $v(x) = e^{\bar{s}(x)}y(x)$ ,  $\bar{s}(x) = \int_a^x s(t) dt$

Then  $y = e^{-\bar{s}}v$  and  $y' + s(x)y = e^{-\bar{s}}v'$ . The Rayleigh quotient becomes:

$$R[v] = \frac{\int_a^b \tilde{p}(x)(v')^2 dx + \int_a^b Q(x)v^2 dx}{\int_a^b \tilde{\omega}(x)v^2 dx}$$

where the transformed coefficients are:

$$\tilde{p}(x) = p(x)e^{-2\bar{s}(x)}, \quad Q(x) = q(x)e^{-2\bar{s}(x)}, \quad \tilde{\omega}(x) = \omega(x)e^{-2\bar{s}(x)}$$

## Key Observation

This is now a **self-adjoint** Dirichlet problem in  $v$  — the minimax principle applies. Note:

$$\frac{Q(x)}{\tilde{\omega}(x)} = \frac{q(x)}{\omega(x)}$$

The exponential factors cancel — bounds depend only on the original coefficients  $q$  and  $\omega$ .

# Eigenvalue Bounds

## Theorem (Lower Bound)

Let  $f \in C[a, b]$  with  $0 < f(x) \leq \tilde{p}(x)$  and  $\tilde{\omega}(x)f(x) < c$  on  $[a, b]$ . Define  $m = \min_{x \in [a, b]} \frac{q(x)}{\omega(x)}$ . Then:

$$\lambda_n \geq \frac{n^2 \pi^2}{c \left( \int_a^b \frac{1}{f(x)} dx \right)^2} + m$$

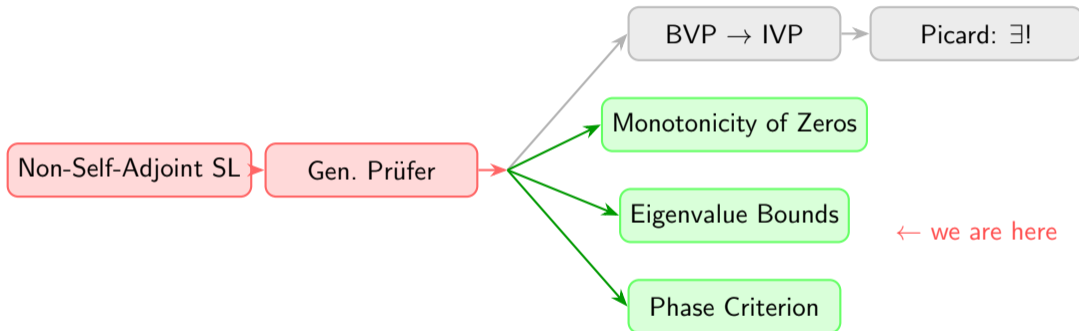
## Theorem (Upper Bound)

Let  $h \in C[a, b]$  with  $h(x) \geq \tilde{p}(x)$  and  $D = \min_{x \in [a, b]} \tilde{\omega}(x)h(x) > 0$ . Define  $M = \max_{x \in [a, b]} \frac{q(x)}{\omega(x)}$ . Then:

$$\lambda_n \leq \frac{n^2 \pi^2}{D \left( \int_a^b \frac{1}{h(x)} dx \right)^2} + M$$

Bounds are **explicit** in terms of the coefficients  $p, q, s, \omega$  — no need to solve the differential equation.

# Road Map



# The Phase Criterion

## Theorem

$\lambda$  is an eigenvalue of the boundary value problem **if and only if**:

$$\theta(b; \lambda) = n\pi \quad \text{for some } n \in \mathbb{N}$$

where  $\theta(x; \lambda)$  solves the phase equation with  $\theta(a; \lambda) = 0$ .

The corresponding eigenfunction is given **explicitly** by:

$$y(x; \lambda) = r(x; \lambda) \sin \theta(x; \lambda)$$

Recall the **Sturm oscillation theorem** fails for our non-self-adjoint problem. However the phase criterion **recovers** this information:

- $\theta(b; \lambda_n) = n\pi$  labels the  $n$ -th eigenvalue
- $y_n$  has exactly  $n - 1$  interior zeros

This is the **non-self-adjoint analog** of the Sturm oscillation theorem.

## Phase Criterion: Proof (Forward Direction)

**Claim:** If  $\lambda$  is an eigenvalue  $\implies \theta(b; \lambda) = n\pi$  for some  $n \in \mathbb{N}$ .

Suppose  $\lambda$  is an eigenvalue with nontrivial solution  $y$  satisfying  $y(a) = y(b) = 0$ .

Write  $y$  in Prüfer form  $y(x) = r(x) \sin \theta(x)$  with  $r(x) > 0$ . Then:

$$y(a) = 0 \implies r(a) \sin \theta(a) = 0 \implies \sin \theta(a) = 0 \implies \theta(a) = 0$$

$$y(b) = 0 \implies r(b) \sin \theta(b) = 0 \implies \sin \theta(b) = 0 \implies \theta(b) = n\pi$$

Since  $y$  is nontrivial,  $n \neq 0$ . By convention  $n \in \mathbb{N}$ .  $\square$

## Phase Criterion: Proof (Backward Direction)

**Claim:** If  $\theta(b; \lambda) = n\pi$  for some  $n \in \mathbb{N} \implies \lambda$  is an eigenvalue.

Start with  $\theta(a; \lambda) = 0$  and solve the phase equation forward to  $b$ . Define:

$$y(x; \lambda) = r(x; \lambda) \sin \theta(x; \lambda)$$

Check boundary conditions:

$$y(a) = r(a) \sin(0) = 0 \checkmark$$

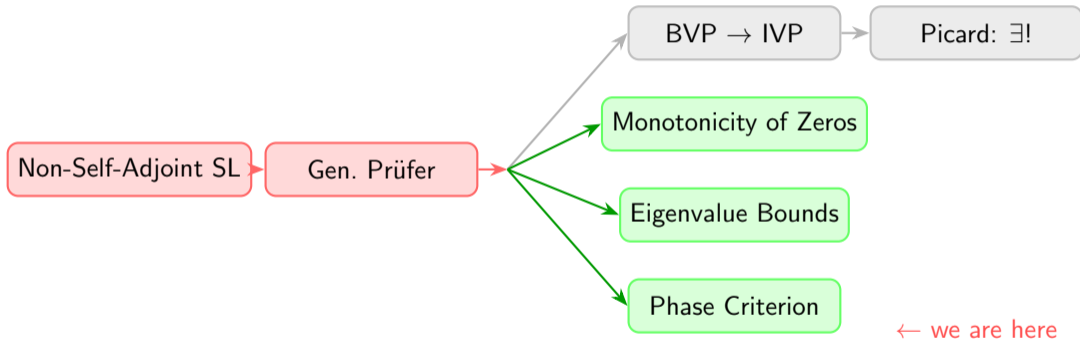
$$y(b) = r(b) \sin(n\pi) = 0 \checkmark$$

$y$  is nontrivial because  $\theta$  varies continuously from 0 to  $n\pi$  with  $n \geq 1$  — there exist points  $x \in (a, b)$  where  $\sin \theta(x) \neq 0$ , and since  $r(x) > 0$ :

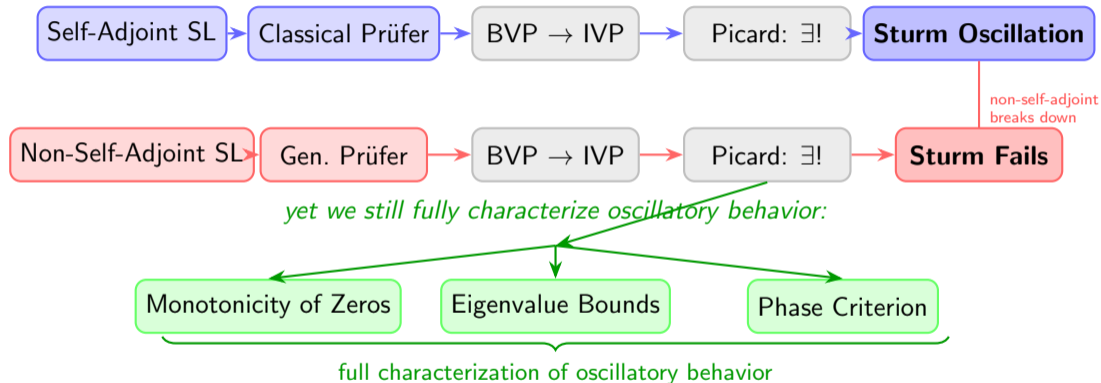
$$y(x) \neq 0 \text{ at those points}$$

Therefore  $\lambda$  is an eigenvalue with eigenfunction  $y(x; \lambda) = r(x; \lambda) \sin \theta(x; \lambda)$ .  $\square$

# Road Map



# The Complete Picture









## Current Work

- **Difference equations** — Prüfer framework in the discrete setting (Çetinkaya with students)
- **Time scales** — unified framework encompassing continuous and discrete settings (Çetinkaya and Cuchta)
- **Difference equations with non-uniform step size** — ongoing work with REU (funded by MAA-NSF) undergraduate students (Bandyopadhyay with students)

## Future Directions

- **Numerical implementation** — computational validation of eigenvalue bounds and convergence analysis for eigenvalue computation in non-self-adjoint settings (completely open)

# References

-  H. Prüfer,  
Neue Herleitung der Sturm-Liouvilleschen Reihenentwicklung stetiger Funktionen,  
*Mathematische Annalen*, 95(1):499–518, 1926.
-  J. C. F. Sturm,  
Mémoire sur les équations différentielles linéaires du second ordre,  
*Journal de Mathématiques Pures et Appliquées*, 1:106–186, 1836.
-  A. Zettl,  
*Sturm–Liouville Theory*,  
American Mathematical Society, 2005.
-  E. A. Coddington and N. Levinson,  
*Theory of Ordinary Differential Equations*,  
McGraw-Hill, 1955.
-  E. B. Davies,  
*Linear Operators and their Spectra*,  
Cambridge University Press, 2007.
-  S. Bandyopadhyay, F. A. Çetinkaya, and T. Cuchta,  
Prüfer Transformation and Spectral Analysis for a Sturm–Liouville-Type Equation,  
*Boletín de la Sociedad Matemática Mexicana*, accepted, 2026.

# Thank You!

Questions and Discussion

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